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THE ART OF QUICK RECKONING

KARL MENNINGER

1875

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CALCULATOR'S CUNNING

THE ART OF QUICK RECKONING

By KARL MENNINGER

TRANSLATED FROM THE
TENTH REVISED EDITION

BY

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FOREWORD

An arithmetical trick is a device for solving a problem more easily and surprisingly than by ordinary methods. While one person will see behind it the broad and wonderfully intricate world of numbers, another will be satisfied by his delight in the trick itself. This book will serve both kinds of student. It will teach the beginner to master a series of useful methods and above all to look at the numbers *before* doing any calculation, in order to use the tricks appropriate to them; for the advanced student, and for anyone who enjoys numbers, it will afford many pleasant glimpses into the land of numbers, and stimulate him to a voyage of discovery.

Numbers have a certain magic. It is no accident that their wonderful properties have grown into the theory of numbers, 'the queen of mathematics'. Those relations which the schoolboy knows as arithmetical tricks belong to elementary number-theory.

Why this rather contemptuous term for sensible arithmetical devices? One reason is that they are nameless. There are no concise terms for the different examples, numbers, and methods, so that only the initiated learn these secrets, and the layman remains bewildered by them. For this reason, and so that they may remain alive, particularly in schools, these arithmetical devices will often be described by short, picturesque names and rules of thumb. A trick is killed off by a long-winded description.

Machine-calculation today stunts the growth of proper calculation, and therefore of number-sense, just as the motor-car stunts the growth of legs. In management and banking circles this is loudly lamented. As for calculating-machines, every teacher has experienced something of this sort: a sixth-former, faced with the exercise 2 × 2, gets out his slide-rule and gives the 'exact' answer 3·95. Every teacher also knows how efficiently the slide-rule can be combined with a well-developed facility in arithmetic.

This book, to which the introduction serves as a preliminary guide, will help to arouse and develop this facility, and to enable the reader to calculate more skilfully and enjoyably. It will also be a reference book for all teachers (including parents), in which they can learn the possibilities and methods of profitable arithmetic.

To calculate enjoyably—that sounds like aiming too high. But number-sense is more widely spread among the population than one would think, and often needs only a favourable impulse to awaken it.

FOREWORD

This book has now been in existence for a generation: the first edition was published in 1931. It has in this time made many friends, who have been grateful for the devices themselves, or for the pleasure obtained from acquaintance with numbers, or both. May it gain new friends now that it has been published for the tenth time, in new dress!

KARL MENNINGER

Bergstrasse, Heppenheim
Autumn, 1960

TRANSLATOR'S NOTE

The word 'unit' is used in the book in two somewhat different senses. Firstly, there is the usual sense of the 'units' place in a number; secondly, it occurs in the phrase 'unit-boundary', meaning a number of the form 10, 100, 1000, ... (The author deliberately avoids using the phrase 'power of 10' until late in the book, so I thought I should too.) There is, I hope, no confusion in this dual use of unit.

It was difficult to know what to do about the monetary examples in the book. German currency, like that of most other countries, is based on the decimal system, so it was impossible to use English currency. In the end, I decided to use dollars ($), as these are fairly familiar to English-speaking people. The examples are, of course, applicable to any other currency which is based on the decimal system.

In examples where distance is involved, I have retained metres (m) and kilometres (km).

E. J. F. Primrose

1875

CONTENTS

STATE INSTITUTE OF EDUCATION BANIPUR

	Page
FOREWORD	5
ABBREVIATIONS	12
INTRODUCTION	13
ANSWERS	124
INDEX	128

I. TESTING A CALCULATION §§1–15

	Sections
Examining a calculation	1
THE NINE-TEST Cross-sum of a number, 2 The test for multiplication, 3 A further example, 4 Is the answer right if the test holds? (Errors of position, 9-errors), 5 Disadvantage and advantage (casting out 9-packets, 0-factor), 6 Summary, 7 Name and proof of test, 8 The test for division, 9 for addition, 10 for subtraction, 11 Last word on the test (test for powers and roots), 12	2–12
THE ELEVEN-TEST Eleven-remainder, 13 The 11 test, 14	13–14
THE DOUBLE TEST 9–11-test, 15	15

II. DEVICES FOR THE FOUR OPERATIONS: OR ARITHMETICAL TRICKS §§16–88

ADDITION 16–23

Adding up, 16 Ten-packets, 17 Making up complete tens, 18 The steps, 19 Multiplication, 20 Mental addition, 21 Two general remarks about mental arithmetic, 22 An addition trick, 23

SUBTRACTION 24–27

Subtraction by adding up, 24 Audit, 25 Two methods for mental subtraction, 26 Subtraction from a unit-boundary, 27

MULTIPLICATION 28–65

MULTIPLICATION BY PARTICULAR NUMBERS . . 28–38

By 5, 28 To simplify the calculation, 29 By 25, 30 By 75, 31 By 125, 32 By 9 and multiples of 9, 33 By 11 and multiples of 11, 34 By 15, 35 By 37, 36 By 111, 37 Summary, 38

THE MULTIPLICATION TABLES 39

The large multiplication table, 39 Two-digit numbers with like tens, 40 The medium and small multiplication tables (on the fingers), 41

CROSS-MULTIPLICATION 42–48

For two-digit numbers, 42 Two hints, 43 Larger numbers, 44 Cross-multiplication for numbers with more digits (sliding and funnel methods), 45 The trick of the double digit, 46 Double digit with 10-complements, 47 Summary, 48

COLUMN MULTIPLICATION 49

For T-like numbers with and without 10-complements, 49

MULTIPLICATION OF NUMBERS NEAR A BOUNDARY 50–55

Three cases, 50 Numbers below a boundary, 51 Numbers above a boundary, 52 Numbers enclosing a boundary, 53 Mirror-numbers, 54 If boundary is not a unit-boundary, 55

DEVICES FOR WRITTEN MULTIPLICATION . . . 56
Factor near boundary, 56 (1) Use of partial answers, 56 (2) Factor with a 1, 56 (3)

FACTOR MULTIPLICATION 57
Splitting up a number into factors and recombining, 57

SQUARES 58–64
Survey of the method, 58 Written method for two-digit numbers, 59 'Funnel' for numbers of three or more digits, 60 Squares of numbers from 1 to 100 mentally (beginning-, middle-, end-squares; a general rule), 61 Particular squares (5-, 1-, 25-squares, squares of numbers near a boundary), 62 Three more particular types of square (for numbers from 400 to 600, for numbers consisting of threes or nines, for 0-mirror numbers), 63 Neighbouring squares, 64

MULTIPLICATION IN RETROSPECT 65

DIVISION 66–88

DIVISION BY PARTICULAR NUMBERS 66–71
By a unit-boundary 10, 100 and so on, 66 By 5, 67 By 25, 68 By 125, 69 Short division, 70 Further methods for division, 71

FACTOR-DIVISION 72–75
The general idea, 72 Factor-division with remainder, 73 If the divisor splits up into several factors, 74 Factor-division with the 25- and 125-tricks, 75

RULES FOR DIVISIBILITY 76–79
Splitting a number into factors, 76 Simplest rules for divisibility (for 2, 4 & 8, for 3 & 9, for 5 & 25, for 11), 77 Divisibility by 7, 11, 13 (the fabulous number 1001), 78 Two general rules for divisibility for any divisor (*A*. Method T+*h* U with auxiliary number *h*, rules for 7, 13, 17, 19; *B*. Method of complement-remainders), 79

BOUNDARY DIVISION 80–87
Essence, 80 Case of divisor below boundary, 81 Division cross, 82 Simplification for experienced calculators, 83 Case of number exceeding the boundary, 84 Division by a number near unit-boundary, 85 Other kinds, 86 Combination of factor- and boundary-division, 87

RETROSPECT 88

III. ESTIMATES §§89–92

Estimate determines order of magnitude, 89 Boundary estimates for decimals, and round numbers in powers of 10, 90 Estimation with tricks, 91 Estimation by stem-fractions, 92

IV. PERCENTAGES §§93–99

Key equation, 93 Idea of percentages, 94

THE FUNDAMENTAL TYPES OF SUM 95–99
Calculation of percentage-value, 95 Percentage-ratio, 96 Working out the percentage ratio, 97 Capital value, 98 Summary, 99

CONCLUSION § 100

Catching the wren, 100

ABBREVIATIONS

C, CS	Cross-sum
DD	Double digit
ER	Eleven-remainder
H	Hundred(s)
H Th	Hundred thousand(s)
NR	Nine-remainder
T	Ten(s)
Th	Thousand(s)
T Th	Ten thousand(s)
U	Unit(s)

≏ approximately equal

1875

INTRODUCTION

'It's all numbers: how can you "read" such a book?' If you begin blindly at the beginning, it may happen that you soon get stuck in an impenetrable thicket of numbers, instead of wandering through the forest of numbers enjoyably and profitably. However, you will succeed if you note the following remarks.

On the composition of the book. First skim once through the whole book, look over the list of contents, linger here and there, to get an idea of the range which it covers.

The material is divided into sections with large characteristic numbers, which appear at the top of each page. In general every section includes rules, examples, and exercises with answers (p. 124).

The composition of the whole book can be seen from the list of contents: the *tests*, the *devices* for the various types of calculation, *estimates*, *percentages*. Individual items can be found quickly by means of the Index. Abbreviations are explained on p. 12.

On the method of working. Which is the best trick to start with? Whichever takes your fancy! Even though the material has a natural order, you need not go strictly from one section to the next. If you have just learnt a trick for multiplication, you can confidently look at a section on division. The best guide is your own interest.

So that a beginner may avoid picking those tricks which demand the most skill, I recommend this **course**:

Nine-test 1–12 (except 8)
Addition 16–22
Subtraction 24–27 (except 26)
Multiplication 28, 30, 34, 35, 39, 51, 52, 56, 62a
    later: 42, 43, 46, 47, 49,1; 53, 54
Squares 58, 59, 61, 62a
Division 67, 68, 70, 72–74; later 78, 80–82, 85, 86
Estimates 89–92
Percentages 93 ff.

Before any calculation make an estimate, *after* it the test. In the book you will learn the test first and estimates at the end (§ 89). This inversion assists the reader: he first learns the test, then does the calculation, and at intervals makes himself familiar with estimates.

Some sections are addressed to certain types of reader.

For the **experienced calculator** (e.g. § 17) there is here and there an application of the rule, which will lead him to the result more quickly and enjoyably. The foundation of all skilled calculation is to work with two-digit, instead of one-digit, numbers. As particular cases, the little-known rules for divisibility (§ 79 *A*, *B*) will appeal to anyone who enjoys numbers.

For **mathematicians** (e.g. § 8) a proof is often carried through algebraically or referred to a formula. Higher mathematics do not appear, so this notice is not a warning sign for those who are not professional mathematicians. On the contrary, all devotees of the mathematical art are entreated to look behind the scenes.

Teachers of arithmetic (e.g. § 26). This includes not only 'official' teachers, but also parents who wish to help their children with a difficulty.

Three working rules must be obeyed carefully.

The important rule. *Don't just read the examples, but work through them.*

The still more important rule. *Not too much at once.* First practise a device which you have learnt by working out the exercises of that section, then in your everyday calculations, so that it becomes second nature, and does not have to be dug out of the memory. This gives a measure of the pace at which you should press on, according to your education and ability.

The third rule. *Pay attention to the remarks on easing the mental burden.* They provide the key to successful arithmetic. You must *understand* a trick, and not just *manipulate* it mechanically. This is the only way to develop an inborn number-sense, which is met with more frequently than one would think even in those who are not bright and clever.

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1. TESTING A CALCULATION

§ 1 EXAMINING A CALCULATION

What tools must one have in order to carry it out? Very few: the addition and multiplication tables. One must know, and not have to work out, that $7+8=15$ and $6\times9=54$, for example. All other calculations depend on these tables. Today everyone learns them; hundreds of years ago knowledge of them was a considerable accomplishment.

If we have worked out a sum by means of the tables, it is important to know whether the answer is *right*. How can we establish this? In three ways:

$$\begin{array}{r} 371\times43 \\ \hline 1113 \\ 1484 \\ \hline 15953 \\ \hline \end{array}$$

(1) **By working it out again,** as it stands. However, it is quite easy to repeat a mistake, without noticing it.

It is better to reverse the sum and work out 43×371. This will usually show up a mistake, but it is hard work and boring. Anyone, particularly a child, is glad to get a sum finished, and if he has to test his answer, he would rather do it by an easy method than by long multiplication. This suggests an

(2) **Estimate,** by means of the rule: work in round figures! So instead of 371×43 we use round figures: $400\times40=16000\approx15953$. We cannot thereby say that our answer is exactly right, but that either *it can be right* because its *order of magnitude* is right (15953 is roughly 16000; both are in the ten thousands, briefly T Th) or *it is clearly false*, because its order of magnitude is wrong. For example, if we had misplaced the second partial product 1484, the answer could have been 2597 (in round figures 3000 instead of 16000, and so Th instead of T Th), although we had made no mistake in the actual multiplication.

The method of estimation is undervalued, not only by beginners: mature students constantly give in an unchecked answer to a long calculation, when a quick and easy estimate would show the answer to be false.

So no calculation without estimation!

Later (§§ 89 ff.) fuller details about estimates will be given. This does not mean that the reader should not make himself familiar with the method before this.

The third method of demonstrating the correctness of a calculation is

(3) **The Test.** This performs invaluable service, particularly in multiplying and dividing; it is simple and fascinating, and together with estimates gives a firm foundation for our calculations. We will now learn about it.

☆ ☆

The Nine-Test

§ 2 THE CROSS-SUM (CS) OF A NUMBER IS THE *SUM OF ITS DIGITS*

Examples C(72)—read 'cross-sum of 72'—is $7+2=9$.

$C(10)=1$ $C(205)=2+0+5=7$
$C(59)=14$ $C(78935)=32$

The last two CS are *two*-digit numbers, whose CS can again be taken, to give *one*-digit numbers:

$C(59)=14;\ C(14)=5$ $C(78935)=32;\ C(32)=5$

Convention: we shall take the CS of a number to be the *final one*-digit number. Thus $C(987886)=1$.

Exercises Find the CS of

(*a*) 905, 901, 37055. (*b*) 11, 119, 1919, 91199. (*c*) 275, 81634, 594027.
If you are observant in your calculations, you will note something remarkable which will be discussed later (§ 6(*b*1)).

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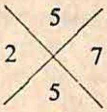
§ 3 THE TEST

We shall test the accuracy of the sum $371 \times 43 = 15953$.

$$\frac{371 \times 43}{15953}$$



Proceed like this: put a cross beside the sum; it forms four regions, left, right, upper, lower. Then

(1) In the *left* region write the CS of the first number 371, $C(371)=2$;

(2) in the *right* region correspondingly $C(43)=7$.

(3) Now multiply the two CS together, $2 \times 7 = 14$; form the CS, $C(14)=5$, and write this in the *upper* region.

(4) Now comes the exciting moment of the test. Form the **CS** of the answer, C(15953) = 5, and put it in the *lower* region.

The upper and lower CS must be equal

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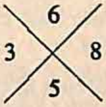
§4 A FURTHER EXAMPLE

Is 4827 × 512 equal to 2470424? We make our test

$$C(4827)=3; \quad C(512)=8; \quad C(3\times 8)=C(24)=6.$$



Now C(2470424) = 5, and the test does *not* hold.

If the test does *not* hold, the answer is *wrong*. So, without doing the sum again, we can easily tell that a mistake has been made. This is the extraordinary feature of our test.

Remember, if the test does not hold, the answer is wrong.

Now that we know there is a mistake in our working, we track it down eagerly: our checking is like an exciting man-hunt. We soon have it by the scruff of the neck: instead of 2470424 it should be 2471424; and now C(2471424) is 6, which passes the test.

Exercises Which of these are wrong? Correct them.

$$\begin{array}{r} 33396\times 421 \\ \hline \underline{14060816} \end{array} \qquad \begin{array}{r} 123456\times 789 \\ \hline \underline{98406784} \end{array} \qquad \begin{array}{r} 41123\times 695 \\ \hline \underline{28584485} \end{array} \qquad \begin{array}{r} 13057\times 897 \\ \hline \underline{11734113} \end{array}$$

Advice: learn gradually to do the test in your head (without the cross).

We now ask, conversely:

☆ ☆

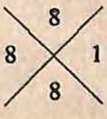
§5 IS THE ANSWER RIGHT IF THE TEST HOLDS?

Unfortunately, we cannot say that it is. Although it is in most cases, there are certain cases in which the test holds but the answer is wrong, and there are two kinds of error which can cause this.

(*a*) **Errors of position** are *not* shown up by our test.

Example

$$\text{Error of position}\quad \begin{array}{r} 485 \\ \underline{208} \\ 3880 \\ \underline{970} \\ \underline{13580} \end{array} \qquad \text{Right}\quad \begin{array}{r} 485 \\ \underline{208} \\ 3880 \\ \underline{970} \\ \underline{100880} \end{array}$$



I have intentionally moved only one place instead of two. The test holds: in both cases the CS of the answer is 8. So our test does not show up errors of position.

What can we do about this? In the case of an error of position the answer will differ from the correct one in the number of digits or the **order of magnitude**, as we shall call it. By this, we mean its highest unit, such as H, T, T Th or H Th. We can find this by means of a simple estimate with round numbers (as in § 1): instead of 485×208 we take $500 \times 200 = 5 \times 20000 = 100000$; the right answer therefore has the order of magnitude H Th and not T Th, as 13580 has. The answer 13580 cannot be right.

Errors in the position of the decimal point are errors of position, and so are *not* shown up by the test, but by the *estimate*. For example, the incorrect sum $0{\cdot}4 \times 0{\cdot}6 = 2{\cdot}4$ passes the test (the correct answer is $0{\cdot}24$).

Remember, if the test holds, estimate the order of magnitude. Then you will avoid errors of position, which the test does not show up. (For estimates, see § 89.)

Exercises Carry out the test and estimate and find any error in

867×2007	743×509	$0{\cdot}17 \times 0{\cdot}206$	389×211
179469	672415	3·50200	82979

(*b*) **Errors of 9 are** *not* shown up by the test.

If you did the last exercise, you came upon something remarkable: (1) the test holds ($2 \times 4 = 8 \mid 8$), (2) the order of magnitude is right ($400 \times 200 = 80000$), but the answer is *wrong*. It should be 82079.

$$\begin{array}{r} 389 \times \underline{211} \\ 389 \\ \underline{778} \\ 82079 \end{array}$$

What has gone wrong?
There is an error of 900.

Why does the test *not* detect this? We see why if we work out the following CS:

$$C(11) = 2; \quad C(119) = 2; \quad C(9119) = 2; \ldots$$

and generally

$$C(n) = a = C(n + 9k), \text{ that is,}$$

the CS of a number does not alter if we add 9 or any multiple of 9, such as 18 ($= 2 \times 9$), 27, 63, 81, . . ., 900, . . ., 27000, ... So $C(52) = 7 = C(6352) = C(5623) = C(815227)$.

Hint. An error of a multiple of 9 can happen in *typing*, if one strikes 0 instead of 9 or conversely; as for example if instead of the right value \$80·75 one types \$89·75.

§6 THIS HAS A DISADVANTAGE AND AN ADVANTAGE

(*a*) **The disadvantage** is that the CS of a number remains the same if the answer differs from the right one by a multiple of 9 (90; 900; 27; . . .). If one miscalculates by such an amount, the test will *hold*.

Now the reader will be indignant: 'It doesn't show up errors of position, though in the end we can deal with these. But if it doesn't show up errors of 9, what good is it?' Don't be too hasty! What good is it?

(1) If the test does *not* hold, the answer is *certainly* wrong.

(2) If we, correctly, assume that any number can occur as an error in our working with the same probability, the chance of an error of 9 is $\frac{1}{9}$ or 11%. These errors, between 1 and 100, are 9, 18, 27, . . ., 99. If we test 100 answers, our test will probably unmask 89 as false and only 11 will go through the net, so small is the proportion of errors of 9. If we did not apply the test, we should not know which of the 100 answers were correct and which were wrong. Which is the more sensible course?

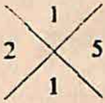
Therefore if you *always* apply the test, you will significantly increase the number of detected errors.

If you want to eliminate errors of 9 and to detect every error with absolute certainty, you should apply, in addition to the 9-test, the 11-test, which we deal with in § 13.

(*b*) **Now for the advantage** in the fact that the CS of a number is *not* altered by adding a multiple of 9.

(1) **The 9-packet**

$$\dot{6}47\dot{3} \times \dot{2}5\dot{7} = 1\underset{\cdot}{6}\underset{\cdot}{6}35\underset{\cdot}{6}1$$



We can find the CS of a number more quickly and certainly if we *cast out* all 9-packets (6+3; 2+7; 6+6+6 = 18 = 2×9; even 1+3+5 if we wish). We put dots over or under the numbers which make them up, so that for the CS we just sum the numbers left over.

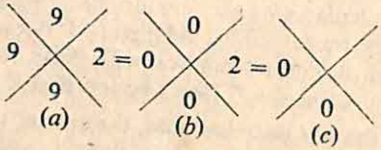
Therefore always cast out 9-packets first, and then form the cross-sum.

Note This rule of thumb teaches you to *examine* the numbers *before* doing the sum: this is most important if you wish to become a good reckoner. But how few people do this. Most people dash blindly at the numbers, and bring up the heavy artillery used against elephants to fire at mice, which they could dispatch instantly if they would only *look*. Only those who **examine numbers** learn to calculate profitably. The 9-packet and the *test* are effective and simple methods for achieving this, and I thoroughly recommend them. It amuses a youngster to see that an enormous number like 4835721 can be carved up so that only the CS 3 remains. If dots have been placed over the numbers to be omitted, they can be eliminated at a glance.

(2) The 0-factor

$$\begin{array}{r} \underline{729 \times 56} \\ 40824 \end{array}$$

Test



(*a*) The test, which you have learnt.

(*b*) Any 9 is replaced by 0: this is permissible in working out the CS.

(*c*) It is *only* necessary to test the *answer*, if the CS has *one* factor 9 (or 0), since it is a 0-factor, as we shall call it. Then $0 \times 2 = 0$.

Note If *one* of the terms is a 0-factor, the answer *must* have CS 9 (or 0). You need not carry out the test completely, but only check the answer.

Exercises Test the exercises of §§ 4 and 5(*a*) by means of 9-packet and 0-factors.

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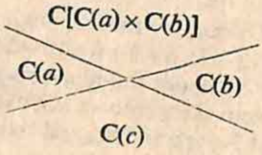
§ 7 SUMMARY

If $a \times b = c$, the test is



Then

(1) If the test does not hold, the answer is wrong.

(2) If the test holds, the answer is generally right, but it can be wrong in the case of:

(*a*) errors of position (or decimal point), which an estimate will show up;

(*b*) errors of 9, where the 11-test helps (§ 13).

Make use of 9-packets and 0-factors!

§ 8 NAME AND PROOF OF THE TEST

Now that we have seen how the test is carried out, we shall see why it is called the 9-test, and on what properties of numbers it depends.

The name of the test. Why is it called the 9-test? Because the CS of a number, for example C(75) = 3, is its *remainder* when the number is divided by 9: $75 \div 9 = 8$, remainder 3.

The *nine-remainder* (NR) of a number is the same as its CS. Show this for the numbers 100; 52; 784; 1000. It is easy to see why this is so.

1 leaves remainder 1 when divided by 9 ($1 \div 9 = 0$, remainder 1).

2 leaves remainder 2; 3 leaves remainder 3, and so on.

10 leaves remainder 1; $20 = 2 \times 10$ leaves remainder $2 \times 1 = 2$; 70 leaves remainder 7.

100 leaves remainder 1 ($100 \div 9 = 11$, remainder 1); $400 = 4 \times 100$ has NR 4.

485 has the following remainder

400	NR 4
80	8
5	5

$$\mathrm{NR}(485) = 4+8+5 = 17 = \mathrm{CS}$$
$$\mathrm{NR}(17) = 8 = \mathrm{C}(17).$$

Instead of referring to the CS of a number, we can speak of its NR.

The proof of the test. Anyone who is not familiar with algebra can omit this section, or just go through the example. Suppose $a \times b = c$.

General case:	Example:
$a \times b = c$	$431 \times 25 = 10775$

Each number can be expressed as a multiple of 9 plus a remainder.

$a = 9m + \mathrm{NR}(a)$	$431 = (9 \times 47) + 8$
$b = 9n + \mathrm{NR}(b)$	$25 = (9 \times 2) + 7$

From now on we shall write R(*a*) instead of NR(*a*).

$a \times b =$	$431 \times 25 =$
$[9m + \mathrm{R}(a)][9n + \mathrm{R}(b)] =$	$[(9 \times 47) + 8][(9 \times 2) + 7] =$
$81mn + 9n\mathrm{R}(a) + 9m\mathrm{R}(b)$	$(9 \times 47 \times 9 \times 2) + (8 \times 9 \times 2) + (9 \times 47 \times 7)$
$+ \mathrm{R}(a)\mathrm{R}(b) =$	$+ (8 \times 7) =$
$9[9mn + n\mathrm{R}(a) + m\mathrm{R}(b)]$	$9[(9 \times 47 \times 2) + (8 \times 2) + (47 \times 7)]$
$+ \mathrm{R}(a)\mathrm{R}(b).$	$+ (8 \times 7).$

Now we form the 9-remainder $\mathrm{R}(a \times b)$ of the product, that is we divide it by 9. Then the remainder is R(*a*)R(*b*), since 9[. . .] is divisible by 9. So

$$\mathrm{R}(a \times b) = \mathrm{R}(c) = \mathrm{R}(a)\mathrm{R}(b), \text{ and}$$
$$\mathrm{R}(431 \times 25) = \mathrm{R}(10775) = \mathrm{R}(431)\mathrm{R}(25).$$

The NR 2 of the answer 10775 is equal to the product of the NR of the factors 431 and 25, namely $8 \times 7 = 56$, or 2. And that is our test!

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For the mathematician. This remainder rule naturally holds not only for the divisor 9, but for every integer, for example 11 or 7. If a, b, c leave remainders $R(a)$, $R(b)$, $R(c)$ on division by d, then if $a \times b = c$, $R(a) \times R(b) = R(c)$. The 'test' therefore *replaces* the numbers by their *remainders*, and carries through the required calculation with *these*, instead of the numbers themselves. Since the divisor is arbitrary, there is not only a 9-test, but also, for example, a 4-test.

Let us check our example $431 \times 25 = 10775$ with the 4-test. On division by 4 the numbers 431 and 25 leave remainders 3 and 1. Then $3 \times 1 = 3$; the answer 10775 leaves remainder 3, and the test holds. The example shows the advantage of the 9-test: the *remainder* is found not through division by 9, but more simply *through its cross-sum*. (See the 11-test in § 13.)

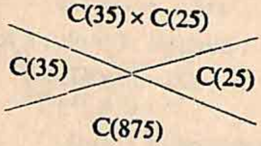
☆ ☆

§ 9 THE TEST FOR DIVISION

The inverse of multiplication $35 \times 25 = 875$ is *division*:

$$875 \div 25 = 35.$$

The usual way of checking this is to multiply the *quotient* 35 by the *divisor* 25 and obtain the *dividend* 875: $35 \times 25 = 875$. However, we can check the accuracy more simply by means of the 9-test. We must have:



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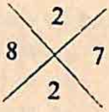


For ease of printing I shall write[*], from now on, $T(8 \times 7 = 56; 2 \mid 2)$, but the reader should always use the cross.

Another example: is $9216 \div 12 = 778$? We test: $T(4 \times 3 = 12; 3 \mid 9)$; the test does *not* hold, so the division is *wrong*. A second attempt gives 768, for which the test holds.

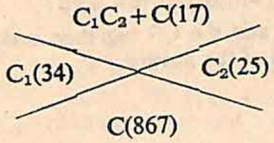
If there is a *remainder*, as in $867 \div 25 = 34$ remainder 17, what then? In this case we make the test thus

$$\begin{aligned} 34 \times 25 &= 850 \\ &\underline{+17} \text{ remainder} \\ &867 \end{aligned}$$

---

[*] In this context, T stands for *test* not *tens*.

Then we work with the cross-sums



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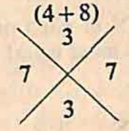

So the test for division runs

Multiply the CS of the quotient by the CS of the divisor; add the CS of the remainder: the resulting CS should agree with the CS of the dividend. Or, more briefly, the

RULE OF THUMB **Quotient × divisor + remainder = dividend.**

Example Test $74803 \div 35 = 2137$ remainder 8.
The test $T(4 \times 8 = 32$ or $5;\ 5+8=13$ or $4 \mid 4)$ holds.

Short cut. Suppose $a \div b = c$, remainder d. If the quotient c or the divisor b is a 0-factor (in the sense of § 6(b, 2)), that is, it has CS 0, the CS of the dividend is equal to the CS of the remainder. Then you need test only the remainder and the dividend.

Example $58474 \div 81 = 721$ remainder 73. Here $b = 81$, which is a 0-factor, $C(81) = 9$ or 0, and $C(73) = C(58474) = 1$.

The proof can easily be found by using § 6(b, 2).

If the test does *not* hold, the answer is *wrong*; if it holds, the answer is right in most cases. Errors of position and errors of 9 will, however, not be shown up. (Do not forget the estimate in round figures.) These flaws should not lead us to underestimate the great value of the test, which is to give an easy way of spotting a mistake. In division particularly is its potency revealed.

Exercises Which of these are wrong? Find the mistakes.

(a) $4867 \div 31 = 156$, (b) $65863 \div 256 = 257$, remainder 71,
(c) $4008 \div 13 = 308$, remainder 5, (d) $2{\cdot}75 \div 15 = 0{\cdot}18$, remainder 5,
(e) $0{\cdot}64 \div 8 = 0{\cdot}08$.

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§10 THE TEST FOR ADDITION

If our test is, through ignorance, rarely used for division, it is still less used for addition. Yet addition is a tiresome business, which often wearies us so much that we get different answers according to whether we start at the top or the bottom. How can we apply our test here?

Find the CS of each number, add the CS together, and form the CS of this sum: this must equal the CS of the answer.

Example

	CS	ER
4858·63	7.	4
27·05	5	10
654·77	2.	5
138·00	3..	6
72·95	5...	-9
424·50	6..	1
8·75	2...	6
43·92	0	-8
209·00	2...	0
6437·55	32	15

Cast out 9-packets! So C(4858·63) = C(88) = 7 and C(43·92) = 0. Write the CS beside the numbers or on a separate piece of paper and strike out the 9-packets from these (i.e. digits with the same number of dots alongside). Instead of 32 just keep 5. C(32) = 5. (The numbers in the ER column are 11-remainders, which we have not discussed yet. See §§ 13 and 14.)

C(6437·55) should be equal to C(32) = 5, but in fact it is 3; the test does not hold, so we have made a mistake. Adding up again gives the right answer 6437·57 with CS 5.

As before, errors of a multiple of 9 will not be shown up. This is discussed again in the 11-test (§§ 13 ff.).

Exercises Write in columns and test

(*a*) 42 + 57 + 68 + 73 + 62 + 54 + 27 + 50 + 89 + 92 + 78 = 662

(*b*) \$6385·75 + 464·50 + 28·73 + 2617·89 + 576·95 + 1469·45 + 730·05 + 796·43 + 345·90 = \$13395·65.

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§ 11 THE TEST FOR SUBTRACTION

The nine-test is rarely used for subtraction. The usual way to check is to add the answer to the number subtracted and see if the sum agrees with the original number.

$$\begin{array}{rr} 4865 & 5 \\ -2973 & 3 \\ \hline 1892 & 2 \end{array} \qquad \text{Check} \quad \begin{array}{r} 1892 \\ +2973 \\ \hline 4865 \end{array}$$

We simply reverse the operation, which for these small numbers is very easy. Compare also § 25.

We can naturally apply our test here too. We write the CS 5, 3, 2 beside the numbers and work with them: 5 − 3 = 2 (or 2 + 3 = 5).

In doing this a slight difficulty can arise.

$$\begin{array}{rcc} 785733 & 6 & 6+9 \\ -39572 & 8 & 8 \\ \hline 746161 & 7 & 7 \end{array}$$

We form the CS 6, 8, 7. However $6-8$ is not 7, and $7+8$ is not 6. The number 6 is too small: the number 8 to be subtracted exceeds it. What is to be done? Since, in working with CS, adding or subtracting 9 makes no difference, we add 9 to 6, and then our test can be carried out.

Add the remainder to the number subtracted, and you should get the original number. Do this with the numbers themselves and also with their CS.

Exercises You can easily make some up for yourself.

Connected with this is a piece of magic which can earn you the reputation of a master calculator in the eyes of those who do not know it. Let anybody choose a number of any length, form another number by altering the order of the digits, and then subtract the smaller number from the larger. He then gives you the digits of the answer in *any* order, omitting *one. You instantly 'work out' this one.*

Example Chosen number 637095
New number 593760

$$\begin{array}{r} 637095 \\ 593760 \\ \hline 43335 \end{array}$$

You are given 3, 5, 4, 3: you say the *missing* one, 3.

What is the trick? It is very simple. The chosen number has a certain CS (3), the new number naturally has the *same* CS (3), $3-3=0$, so the answer must *always* have CS 0, that is, it is 9 or a multiple of 9 (18, 27, . . .). If you are given all the digits except one, you add them together and make the sum up to 9: $3+(5+4)+3=6$, so the number which you give is the one which makes this up to 9, that is, 3.

There is one doubtful case: if the CS is 9, what do you say, 0 or 9?

Example

$$(a)\ \begin{array}{r} 6375 \\ 5376 \\ \hline 999 \end{array} \qquad (b)\ \begin{array}{r} 6375 \\ 3675 \\ \hline 2700 \end{array}$$

In case (*a*) you will be given 9, 9; CS = 9, and so you should say 9.
In case (*b*), if you are given 7, 2, 0, whose CS is also 9, you should say, not 9, but 0.

But unfortunately you cannot tell, so to be on the safe side the 'magician' should say that 0 and 9 are to be regarded as equivalent. For another trick with the CS see § 37.

* * *

§ 12 LAST WORD ON THE TEST

With the following amusing application we leave the nine-test, which we have already learnt to use for all four arithmetical operations. This application is to powers and roots, and has the same advantages and limitations.

Examples

Powers. Is 17^2 equal to 289? We carry out the work with the CS: $C(17^2)=C(8^2)=C(64)=1$; $C(289)=1$, and the test holds.

Is 487^2 equal to 237169? The test T(1; 1) holds, the estimate $500^2=250000$ agrees, so the answer is probably right; only probably, since an error of 9 will not be shown up.

Roots. If someone says $\sqrt{144}=13$, is that right? Test: 144 should be 13^2. The test, $C(144)=0$, $C(13^2)=C(4^2)=7$, does not hold, so the answer 13 is wrong (the correct answer is 12).

The nine-test is therefore a wonderful help in all calculations. Practise it until it becomes second nature.

No calculation without estimate and test!

So much for the nine-test. You can now, if you wish, skip to § 16, and amuse yourself with the actual tricks. However, since safeguards have been mentioned, we shall take this opportunity to discuss another test, which eliminates the blemishes in the nine-test.

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The eleven-test

§ 13 THE ELEVEN-REMAINDER

We have seen in § 8 that the CS is the 9-remainder. Just as we carry out our test with remainders on dividing by 9, so we can use it on dividing by 5, 7, 13, 15, and therefore also with **11**-remainders. The expert reader will see that this is so from the proof in § 8: instead of writing a as $9m+R(a)$, we can express it as

$$a=7s+R(a), \qquad 431=(7\times 61)+4 \text{ or}$$
$$a=11u+R(a), \qquad 431=(11\times 39)+2.$$

Why have we chosen to work with 9 up to now? Because its remainder can so *easily* be found: it is simply the CS of the number.

However the 11-remainder can also be found easily. On division by 11

1 leaves remainder $+1$
100 „ „ $+1: 100 \div 11 = 9$ remainder 1
10000 „ „ $+1: 10000 = 9999 + 1$
and so on.

Therefore all units with an *even* number of zeros or, what comes to the same thing, all units with an *odd* number of places, have 11-remainder + 1. The U counts as 1 place, the H as 3 and the T Th as 5.

Since $400 = 4 \times 100$, $\mathrm{ER}(400) = 4 \times 1 = 4$;
$80000 = 8 \times 10000$, $\mathrm{ER}(80000) = 8 \times 1 = 8$.

What happens to units with an *even* number of places, the T, Th, H Th. . . ? On division by eleven,

10 leaves remainder 10 or $11 - \mathbf{1}$ or $-\mathbf{1}$
1000 „ „ 10 or $11 - \mathbf{1}$ or $-\mathbf{1}$; $1000 = 990 + 10$
and so on.

Just as with 9-remainders, so with 11-remainders, 11 or a multiple of 11 can be added or subtracted, so

$$\mathrm{ER}(10) = \mathrm{ER}(10 - 11) = -\mathbf{1}.$$

Therefore all units with an *even* number of places have 11-remainder $-\mathbf{1}$.

Example $40 \div 11 = 3$ remainder 7; or since $\mathrm{ER}(10) = -1$, $\mathrm{ER}(40) = 4 \times (-1) = -4$. Both remainders are really equivalent, since $-4 + 11 = +7$.

But why use the unfamiliar -4 when $+7$ is equivalent? We see this immediately. We shall find the ER of 475, that is, the remainder when 475 is divided by 11. This can be done conveniently as follows:

$$\mathrm{ER}(5) = 5 \times 1 = 5 \quad \mathrm{ER}(70) = 7 \times (-1) = -7 \quad \mathrm{ER}(400) = 4 \times 1 = 4,$$

so $\mathrm{ER}(475) = 5 + 4 - 7 = 2$. In fact, $475 \div 11 = 43$ remainder 2.

Note We obtain the ER of a number if we subtract the sum of the *even* place digits from the sum of the *odd* place digits; or more briefly according to the

RULE OF THUMB **for the 11-remainder: subtract the 'even' sum from the 'odd'.**

Example $\mathrm{ER}(\underset{.}{4}8\underset{.}{7}2\underset{.}{5}) = 16 - 10 = 6$. To avoid errors *put dots* under the odd places.

Special cases (1) If the ER is less than 0, add 11:
$\mathrm{ER}(195) = -3$ or $-3 + 11 = 8$.
(2) Two consecutive equal digits can be ignored:
$\mathrm{ER}(88) = 0$, $\mathrm{ER}(70755) = \mathrm{ER}(707) = 3$. Why?

Exercises What are the ER of (*a*) 56, 65, 30, (*b*) 217, 271, 2579823? Test your answers by actual division by 11.

Application. If the ER of a number is zero, the number is naturally divisible by 11. To know this can be an advantage in bringing fractions to their lowest terms.

Example 6793545 has ER 0; the number is exactly divisible by 11, the quotient being 617595 (see also § 78).

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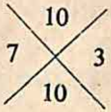
§ 14 THE 11-TEST

now ends *just* like the 9-test, except that, instead of CS(=NR), we use 11-remainders.

Example

$$\frac{557\times 25}{13925}$$

Test

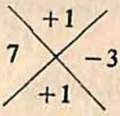


$\mathrm{ER}(557)=7[+5-5]=7$; $\mathrm{ER}(25)=5-2=3$; $7\times 3=21$; $\mathrm{ER}(21)=-1$ or 10; $\mathrm{ER}(13925)=5+9+1-(2+3)=10$, and the test holds.

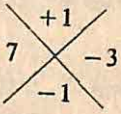
For decimals such as 8·356 we find the ER by regarding the units place as the starting-point for the odd places, and working to the right for the decimal places, so $\mathrm{ER}(8{\cdot}356)=8+5-(3+6)=4$, and $\mathrm{ER}(83{\cdot}56)=9-13=-4$ and $\mathrm{ER}(835{\cdot}6)=4$. We recognise the following rule: a point-shift of 1 place (or an odd number of places) *changes the sign* of the ER, and a point-shift of 2 (or an even number) *leaves it unchanged.* For the 11-test this means: a 1-place error in the point shows up, but a 2-place error does not. Hint: use estimates.

Examples with the same digits as those above:

(*a*) $\dfrac{5{\cdot}57\times 2{\cdot}5}{13{\cdot}925}$



(*b*) $\dfrac{5{\cdot}57\times 2{\cdot}5}{139{\cdot}25}$



(*a*) The test holds. Notice that $7\times(-3)=-21$, $\mathrm{ER}(-21)=-21+(2\times 11)=+1$.

(*b*) The test does not hold. ER(139·25) = 15 − 5 = 10 or −1, so there is probably a 1-place error in the decimal point (the right answer is 13·925).

Should we now say with absolute certainty that if an answer passes the 11-test (as well as the 9-test) it is right? No, because, just as the 9-test does not show up errors of 9, so the 11-test does *not* show up errors of 11. However, of 100 wrong answers probably only $\frac{1}{11}$ will slip through, that is, 9. For the 9-test it was 11 (see § 6(*a*, 2)). This is not a great advantage to claim for the 11-test. But if we use *both* tests *together*, we shall form an extraordinarily fine net, through which hardly any mistake can slip. Before this, however, the reader should use the new test on the same exercises as those for which the 9-test was used in §§ 5(*a*), 9 and 10. In § 10, the numbers in the ER column of the example are 11-remainders. For all correct answers the test must hold, for all wrong answers it will not.

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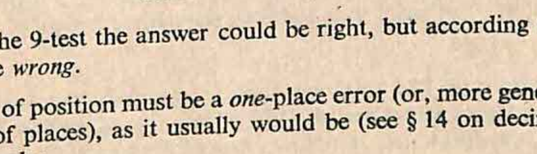
The double test

§ 15 THE 9-11-TEST

The 9-test lets two types of error through, those of position and those of 9.

We choose the example of § 5(*a*), in which the 9-test breaks down because of an *error of position*, and apply the 11-test:

$\dfrac{485 \times 208}{13580}$



According to the 9-test the answer could be right, but according to the 11-test it must be *wrong*.

Note The error of position must be a *one*-place error (or, more generally, an *odd* number of places), as it usually would be (see § 14 on decimals). We take an example:

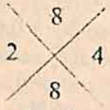
(*a*)	(*b*)	(*c*)
123	123	123
5008	5008	5008
984	984	984
615..	615.	615
615984	62484	7134

(*a*) correct answer, (*b*) *one*-place error, (*c*) *two*-place error. *All* pass the 9-test, and (*a*) naturally passes the 11-test, but so does (*c*). It therefore unmasks only the one-place error in (*b*). Go through this again, to make sure you understand it. The best security against place-errors is always the *estimate*, which shows up an error like (*c*).

Let us test the example of § 5(*b*), whose 9-*error* was not shown up by the 9-test.

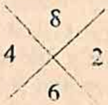
$$\frac{389\times 211}{82979}$$

9-test



holds

11-test



does *not* hold.

The 11-test therefore reveals the answer as *wrong*.

Conclusion. Errors (place-errors and 9-errors) which slip through the 9-test—for which it holds—are caught by the 11-test. So if the 9-test holds, and you want to be certain, carry out the 11-test.

Can you be quite sure? No. Which errors escape the 9-11-test? Only those mistakes which slip through the 9-test and the 11-test, and they are 99-errors (99, 1980, . . .). If you make a mistake such that the answer has an error of 99 or a multiple of 99 you will not find it.

Example Instead of 9589 you get the wrong answer 10084, which is 5 × 99 too much. Then both tests hold.

$\mathrm{C}(9589) = 4 = \mathrm{C}(10084)$: the 9-test holds.
$\mathrm{ER}(9589) = -3$ or $+8 = \mathrm{ER}(10084)$: the 11-test holds.

However, it will probably happen only once in 100 cases that one is wrong by 99; only 11 errors can slip through the 9-test, namely 9, 18, 27, . . ., 99; similarly only 9 errors can slip through the 11-test, namely 11, 22, 33, . . ., 99.

Generally, of 100 wrong answers, 99 are probably caught by the double-test, and only 1 slips through. 99% of all such answers are shown to be false, that is, the double-test lets hardly any errors escape.

For those cases where a correct answer must be unconditionally guaranteed, the double-test is a simple and powerful method, of inestimable value. Then again, think how much more cheerfully we can work if we know we can check that we have worked correctly. Even the mention of an error starts up an exciting hunt for the offender.

Use the test under all circumstances. Usually the 9-test will be sufficient, but to be on the safe side use the 11-test as well.

It is worth mentioning, without going into any details, that the mesh of the net can be made still finer, so that 99-errors cannot slip through. But for everyday calculations a finer net is not necessary.

For the mathematician. The expert will recognise that the double-test works with remainders on dividing by 99. I recommend him to try working with remainders on dividing by 101 or 1001. A divisor is suitable for a test only if the *remainders* can *easily* be found. In connection with some wonderful rules for divisibility I introduce the 37-test in § 79 (*B III*) (at the end).

Exercises Check the answers first with one test, and then, if it holds, with the other.

(*a*) $\dfrac{4605 \times 381}{179595}$ (*b*) $\dfrac{7285 \times 437}{3184645}$ (*c*) $\dfrac{647 \times 872}{564184}$

(*d*) $\dfrac{0{\cdot}2 \times 0{\cdot}4}{0{\cdot}8}$ (*e*) $\dfrac{48{\cdot}63 \times 7{\cdot}5}{3647{\cdot}25}$

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II. DEVICES FOR THE FOUR OPERATIONS, OR ARITHMETICAL TRICKS

ADDITION

§ 16 ADDITION

is for many people an abomination. For one thing, much practice is necessary; but the main source of the trouble is the worry it causes. A column of figures creates a feeling of panic: I have so many numbers to add up that I am bound to make a mistake! This feeling makes one lose one's enthusiasm and courage. And so all the enjoyment is gone. It is as if one were on the bank of a wide stream, from which only a few stones project. One has to cross, and one looks anxiously at the next stone, and shaking in every limb one leaves the stone one is on, after trying first one foot and then the other, to see if one can venture on . . ., and so the whole crossing is a torture, which ends up with a plop in the water.

$$\begin{array}{r} 135 \\ 78 \\ 85 \\ 844 \\ 913 \\ 57 \\ 879 \\ 820 \\ \hline 3811 \end{array}$$

In such a way the anxious reckoner clings to one number after another, his eyes riveted on the next so fixedly as if he would stop there. Everyone knows addition which goes like this: 9 and 7 is 16, and 3 is 19, and 4 is 23, and 5 is 28, and 8 is. . . and 8 is. . . and 8 is 36, and 5 is 41. At least he can count himself lucky not to have lost the 28 in the threefold charge on the 8.

Why does this happen? Because he uses so much energy in hanging on that he has none left for the addition.

What shall be done about ? Hop! Simply hop! But what happens if one falls in the water? We can at least show by our tests that we have fallen, and that is a great comfort.

In order to hop we now learn two tricks.

☆ ☆

§17 TEN-PACKETS

require us to 'keep our head high'. Don't just look at the next number, but at a whole series, and any two numbers which add up to **10** should be put in a packet, whether they follow each other or are separated by other numbers. So, not

$$9,\ +7=16,\ +3=19,\ +4\ldots$$

but

$$9,\ +(7+3)=19,\ +4=23,\ +(5+5)=33,\ +8=41.$$

We have two 10-packets $(7+3)$ and $(5+5)$, and so 4 digits disappear from our column at one stroke, and we really have to add only 3 instead of 7: a considerable lightening of the load, which cannot be overestimated, particularly since the answer is much more certain than it would otherwise be. Just try it, once step by step, and once with 10-packets.

The use of 10-packets makes the work much more exciting: you hunt them down, you are pleased if you search them out, or if a giant series of numbers suddenly falls into 10-packets. That happens much more often than you would think.

The experienced calculator will notice 10-packets of *three* numbers such as (4, 5, 1); (4, 2, 4) or 20-packets such as (6, 6, 8) and (8, 8, 4).

For the teacher. If you have occasion to demonstrate 10-packets to young children, say 10 years old, you will be amazed: suddenly a poor, unhappy child becomes enthusiastic, as though he had picked out a plum, as though a torture had been turned into a pleasure. Also, it is excellent practice for looking at numbers (§ 6).

One thing you should always remember if you are teaching arithmetic: always be cheerful, and introduce a new topic with a smile, not with a frown. We learn best by 'playing', as we do in the early stages of arithmetic.

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§18 MAKING UP COMPLETE TENS

In our example, if I have added 7 and 9, giving 16, I pick out a 4 to make up to 20, a 'complete ten'. A complete T is a resting-place, where one can recover one's breath.

In mental arithmetic a complete T plays a still more important role

$$\begin{aligned}34+28+15+26+45+12&=(34+26)+(28+12)+(15+45)\\&=60+40+60=160.\end{aligned}$$

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§ 19 THE STEPS

are the next thing which we consider in addition:

Never recite the method, but only the answers, the steps.

Do not say '9 and 7 are 16, and 3 are 19, and 4 are 23, and . . . and . . .', since the numbers are already there, the eye sees them, the mind notices them; why then bring the mouth into it? Saying them *breaks* the thread of the calculation. Long before you have said 7 + 8 you have the number 15 on the tip of your tongue. It is a bad habit to keep on reciting the addition and multiplication tables, which one must *know* anyway, as if they were part of the calculation. It is rather like reading 'h + e = he'. It is also like standing rigidly on the stone and summoning up courage to go over to the next one. We work out our example of § 16 in steps like this:

9, 19, 23, 33, **41**; 4, 6, 13, 23, (5 + 1 + 4 = 10), 33, **41**;
and 3 × 8 = 24, 28, **38**.

From the last sequence we learn

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§ 20 MULTIPLICATION,

as the combination of *equal* numbers.

We pick out (example in § 16) the three digits 8, and instead of 8 + 8 + 8 we say 3 × 8. You may be inclined to say 'It's obvious that you do that'—but how many times have you caught yourself overlooking the best possibility? Why? Because you are not used to looking at the whole column of figures.

However, if you consider from now on our four hints, you will be surprised to see how quickly you learn to hop, and how much more certainly and enjoyably you add up than you did previously. First try some exercises by your usual method, and then for comparison or for pleasure hop through them again. Hopping certainly does not lead to carelessness: on the contrary, the number-sense works much more effectively if you let it have its head than if you are continually drawing in the reins. You always have the 9-test as a safeguard.

So remember for addition

(1) **10-packets**
(2) **Complete tens**
(3) **Steps**
(4) **Multiples**

There is always the test of § 10.

Examples from § 10.

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§21 MENTAL ADDITION

In mental addition it is generally just a question of two numbers: 38+56, for example. This is usually done: 38+50=88, +6=94. It goes more easily according to the rule: first the units, then the tens. So 38, 44, **94**. Naturally you give only the steps.

However, the experienced calculator grasps the numbers much more quickly: he must give the sum of 2-, 3- or 4-digit numbers *immediately* according to the

RULE OF THUMB **Split up the numbers and add them from the left.**

I cannot sufficiently emphasise this splitting up of a number for mental arithmetic. In schools this should be taught with all possible emphasis. To start with you can split the number up by means of a small stroke at the top. These examples will demonstrate the method.

25	I split	2'5	
+34	up into	3'4	and you see immediately 5'9, that is, 59.
143	14'3		
+74	7'4	21'7	'twenty-one seven'.
352	is child's	35'2	
+126	play	12'6	
		47'8	'forty-seven eight'.

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If the sum of the units exceeds ten,

47	I *see* this, and add one T (it cannot be more), and so **8'5**.
+38	

So you can write the answers to these examples from the *left*

48	157	429	237	$14·78
86	86	168	656	$32·56
13'4	24'3	59'7	89'3	$47·34

In written work you *see* the numbers split up, but in mental arithmetic you *say* them split up: 15'7+8'6=24'3.

Anyone who has practised this a little will be surprised at the simplification which splitting up provides; we shall often meet it later on (see §26 (1)). Apart from this, it teaches us to look at the whole problem *before* doing any calculation, and this is most important. How often is a simple problem mistakenly added from the units column backwards, when a quick glance would give the answer straight away.

$$(a)\quad \begin{array}{r} 4\,237 \\ +\,1\,948 \\ \hline \end{array} \quad \text{and} \quad (b)\quad \begin{array}{r} 42\;37 \\ 19\;48 \\ \hline 61\;85 \end{array}$$

While in (*b*) the groups 42 and 19, and 37 and 48, can quickly be added mentally, and the answer written down from left to right, the same example is difficult to do in the form (*a*), and can only be solved in the old way, backwards from right to left.

Note on language No doubt many people will regard the expression of 4237 in the form 'forty-two (hundred) thirty-seven' as bad English. Now this it is certainly not, because we speak, for instance, of the year 1950 in this way. What is more important is to split up a number in preparation for calculation, and this has nothing to do with language. The answer can then be expressed in the familiar way.

* * *

§ 22 TWO GENERAL REMARKS ABOUT MENTAL ARITHMETIC

will now be made.

(*a*) **Semi-mental arithmetic.** Every method of calculation consists of two mental processes: *calculation* and *retention* of the results already obtained.

Now it is clear that a mind which is being loaded with an ever-increasing burden cannot work as freely as one which is unburdened. It is therefore an important condition of mental arithmetic that the burden of retention should be removed, and so the partial answers should be *written down*. You will be surprised how quickly you can work. Often the retention is more difficult than the calculation. In our further work we shall have many occasions to use this 'semi-mental arithmetic', as we shall call it, either for the whole problem or at the beginning.

(*b*) **The carry-over.** Teachers often require children to carry over in their heads too early. This makes arithmetic more difficult for them than grown-ups are inclined to believe. But if a little chap can write down 4 in the tens column when he has 45 units, he can take a breath, and the rest of the calculation will go more easily. Compare § 70.

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§ 23 AN ADDITION TRICK

for the special case where successive terms differ from each other by the same amount.

$$1+3+5+7+9+11=\,?\quad\text{or}\quad 1+3+5+7+9+11+13=\,?$$

The difference is in both cases always 2. Call the sum s, the difference d, the first term a, the last term z, and the number of terms n. Then we have the

RULE **Multiply the sum of the first and last terms by half the number of terms. As a formula:**

$$s=\tfrac{1}{2}n(a+z).$$

In our examples:

$$s=3(1+11)=36; \quad s=\tfrac{7}{2}(1+13)=7\times 7=49.$$

Proof Form pairs of numbers, the first and last, the second and last but one, and so on; the sums are equal. How many are there? The mathematician calls such series of numbers 'arithmetical progressions'.

Examples

$$15+19+23+27+31+35=3(15+35)=3\times 50=150.$$

$$132+140+148+156+164=\tfrac{5}{2}(132+164)=5\times 148=740.$$

As we see, if the number of terms is *odd*, the answer is simply the *middle term* multiplied by the *number of terms*: 148 × 5.

Exercises (*a*) The sum of all the numbers from 1 to 20; (*b*) the sum of the even numbers from 1 to 20; (*c*) $68+73+\ldots+93$; (*d*) $153+168+183+198+213$.

SUBTRACTION

§ 24 SUBTRACTION

should always be done *by adding up* (the so-called Austrian* way):

$$\begin{array}{r} 6305 \\ -4289 \\ \scriptstyle 1 \\ \hline \ldots 6 \end{array}$$

(*a*) 9 and *six* is fifteen. Say 6 and *write down* 6 simultaneously; then say 'fif-teen', that is with a slight pause after the 'fif-' so as to write the carry-over 1 under the 8 when you say '-teen'.

Once more: 9 and six (write) is fif-teen (write). The example now looks as it does above.

(*b*) 1, 9 and *one* is *ten* (carry over).

(*c*) 1, 3 and *zero* is 3.

(*d*) 4 and *two* is 6; the answer is 2016.

Thus we do not subtract, but add, which is easier. However, this method has an even greater advantage over the old method: 9 from 15 is 6 . . . ; the ten must be 'borrowed' from the 3 H, and in the tens column we now have not 0 but 9, which is difficult to remember. 8 from 9 is 1; 2 from 2 is 0 and 4 from 6 is 2. The number 6,3,0,5 in front of us is transformed into a number 6,2,9,5 which is *not* in front of us and which must continually be *retained.*

Certainly the new way demands strict agreement between speaking and writing. Anyone who is too easy-going will, in the step 9 + 6 = 15, put down the 5 instead of the 6, a common error among school-children, who tend to fix on the last number.

Exercises

$$\begin{array}{r} 457288 \\ -53496 \\ \hline \end{array} \qquad \begin{array}{r} 2{\cdot}70342 \\ -1{\cdot}09684 \\ \hline \end{array} \qquad \begin{array}{r} 546{\cdot}037 \\ -489{\cdot}9764 \\ \hline \end{array}$$

For the experienced calculator it is good practice to find the remainders in a division sum by immediate subtraction.

$$\begin{array}{l} \underline{37958} \div 427 = 8 \ldots \\ \;379 \quad {\scriptstyle 26} \end{array}$$

(*a*) 8 × 7 = 56, + 9 = 65; 9 is the remainder, and I write the carry-over 6 as a small figure underneath.

(*b*) 8 × 2 = 16, + 6 = 22, + 7 = 29. Remainder 7, carry-over 2.

(*c*) 8 × 4 = 32, + 2 = 34, + 3 = 37. Remainder 3.

Anyone who can calculate accurately will enjoy this method, particularly if he *writes down* the carry-over figure. Checking is certainly more difficult,

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**Translator's footnote:* there is no well-known term in English for this method.

since the partial products are not written down; moreover a partial product which appears for a second time has to be worked out again. So I do not recommend this method for beginners. Nevertheless the reader should try the exercises in § 73 in this way.

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§ 25 THE AUDIT*

in *one* stroke, namely the subtraction of *several* amounts from one, is a valuable application of the method of adding up.

From a sum of $635·72 several expenses are to be subtracted: $49·85; $72·50; $9·15; $14·45. We add these expenses together and immediately find the difference between their sum and $635·72. We separate this amount from the others by means of a line.

$$\begin{array}{rr} 635{\cdot}72 & 5 \\ \hline -\ 49{\cdot}85 & 8 \\ -\ 72{\cdot}50 & 5 \\ -\ 9{\cdot}15 & 6 \\ -\ 14{\cdot}45 & 5 \\ {\scriptstyle 2\,3\,2\,2} & \\ \hline 489{\cdot}77 & 8 \\ \cline{2-2} & 5 \end{array}$$

The units total 15. I have now reached the line: how can I get over it to the 2? I count up from 15 to the *next* number which ends in 2, which is 22. Now it goes as in § 24: 15 and seven is twenty-two (carry over 2); 2, 12, $20+\mathbf{7}=27$; 18, 20, $26+\mathbf{9}=35$; 10, $15+\mathbf{8}=23$; $2+\mathbf{4}=6$. The answer is $489·77. I have made profitable use of adding up (§§ 17 ff.).

The test holds: the sum of all the CS agrees with the CS 5 of the original amount. Or without the 9-test: add the remainder to the expenses and you get the original amount; the carry-over figures are the same as those written down during the subtraction.

Exercises Write the amounts under one another, but subtract across also.

(*a*) 4287·25 − 312·45 − 82·95 − 1064·20 − 450·00

(*b*) $3125·00 − 24·70 − 857·75 − 300·84 − 0·27 − 54·92 − 89·11

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§ 26 TWO METHODS FOR MENTAL SUBTRACTION

which the experienced calculator should adopt, namely subtraction with *split* numbers and with *100-complements*.

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* *Translator's footnote:* I have translated this literally, as there is no equivalent term in English.

(1) **Splitting up** (as in addition, § 21)

$$\begin{array}{rlrrr} 88 & \text{becomes easier} & 8'8 & 127 & 435\\ -\underline{32} & \text{like this} & \underline{3'2} & \underline{43} & \underline{72}\\ 56 & & 5'6 & 8'4 & 36'3\end{array}$$

If the units exceed ten, subtract 1 from T.

$$\begin{array}{rlrrr} 83 & \text{becomes} & 8'3 & 132 & 676\\ -\underline{35} & & \underline{3'5} & \underline{78} & \underline{349}\\ 48 & & 4'8 & 5'4 & 32'7\end{array}$$

Always write from the *left*. To begin with, split the numbers by means of a stroke at the top, but later a 'glance' or an ingenious enunciation (such as 132 as thirteen-two) will be sufficient.

(2) **The 100-complement** of a (two-digit) number is the number which makes it up to 100.

$$67+\mathbf{33}=100;\quad 42+\mathbf{58}=100;\quad 12+\mathbf{88}=100.$$

For the teacher of arithmetic. Children are taught to be familiar with numbers not only by means of the multiplication table, but still more by means of the **100-complement**, the **double** and the **half** of all two-digit numbers. You say '100-complement' and then the ball goes to and fro. The teacher throws 37 to a boy and he hurls back 63; faster and faster until the head reels! Similarly for the double: 84 out, 168 back! And the half: 176 out, 88 back! By this kind of play the number-sense will be awakened and strengthened, and at the same time the young calculator gains a valuable store of number-bonds, which will be of great service in all calculations.

Every calculator must know 100-complements, doubles and halves by heart, and not have to work them out first. It is better still to master **1000-complements** (387 out, 613 back!).

If you add to this the **squares** of numbers from 1 to 25 (§ 61(*A*)), by heart, and practise all these in the lower forms, you can bring up children from 10 years old to be surprisingly experienced calculators. Above all, they overcome their fear of numbers and gain courage to tackle harder exercises more bravely. Just try it once, and see how the children enjoy it!

Now for subtraction by H-complements (I write H- instead of 100-complements, as we shall make up, not only to 100, but also to 200, 300, . . .).

$$\begin{array}{r} 315\\ -\underline{176}\\ \mathbf{139}\end{array}$$

Subtract 200, =115, and add the 100-complement of 76, thus 115+24=**139**; or first add 24 (339), then subtract 200 (139).

$$\begin{array}{rlrl} 4'26 & 4-3=1 & 7'64 & 7-3=4\\ -\underline{2'43} & 26+57=83 & \underline{2'89} & 64+11=75\\ 1'83 & & 4'75 & \end{array}$$

RULE **Subtract the next complete H and add the 100-complement.**

With practice 3 and even 4 digit numbers can be subtracted from each other in this way.

Exercises Work out these by splitting up and by H-complements.

$$\begin{array}{r}142\\-\underline{87}\end{array}\quad\begin{array}{r}356\\-\underline{98}\end{array}\quad\begin{array}{r}423\\-\underline{67}\end{array}\quad\begin{array}{r}654\\-\underline{379}\end{array}\quad\begin{array}{r}8472\\-\underline{786}\end{array}$$

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§ 27 SUBTRACTION FROM A UNIT-BOUNDARY

such as 100, 1000, 10000, . . ., is simply done:

$$\begin{array}{r}10000\\-\underline{7345}\end{array}$$

Working from left to right, subtract each digit except the final one from 9 and the last from 10.

So you can write down the answer from the left: 2655. Such exercises arise in giving change for notes. If anyone pays a bill of \$48·37 with a 100 dollar note his change is \$51·63.

Note on language We always write \$51·63, but we say '51 dollars 63', not 'fifty one point six three dollars'. Quite apart from the fact that no business man would say the latter, it would destroy the convenient splitting of the number.

Exercises (*a*) 10000 − 6789; (*b*) 100000 − 56837; (*c*) \$10 − \$7·36.
(*d*) Which trick can you find for \$1000 − \$605·89, and for \$600·00 − \$345·68?

MULTIPLICATION

Multiplication by particular numbers

§ 28 BY 5

$$12\times 5=60;\quad \text{but } 5=\tfrac{10}{2}, \text{ so } 12\times 5=12\times \tfrac{10}{2}$$
$$\text{or more cunningly } \tfrac{12}{2}\times 10=6\times 10=60.$$

RULE **To multiply a number by 5, multiply half of it by 10, that is, halve it and append a zero.**

Thus

$$\begin{array}{lc} 5\times 126=(63),\ \mathbf{630} & 638574\times 5 \\ 5\times 458=(229),\ \mathbf{2290} & \overline{319287 \mid 0} \end{array}$$

What happens if the number is *odd*, so that it cannot be halved exactly, e.g. 17×5? We still halve it, $17\div 2=8$ remainder 1, and put **5** at the end; we leave out the remainder: 85. Or like this: $17\div 2=8\tfrac{1}{2}$; the integral part is 8, then 5. So **for odd numbers**

take the integral part of half the number, and append 5 (instead of 0).

$$\begin{array}{cccc} 167\times 5 & 543\times 5 & 63788\times 50 & 63789\times 50 \\ \hline 83 \mid 5 & 271 \mid 5 & 31894 \mid 00 & 31894 \mid 50 \end{array}$$

To multiply **by 50** we multiply by 5 and 10, so instead of 0 or 5 we append 00 or 50.

Exercises Multiply by 5

(*a*) 72, 126, 448, 836, 5468; (*b*) 87, 143, 275, 4833, 640571

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§ 29 TO SIMPLIFY THE CALCULATION

I add two remarks.

(*A*) Group-Vision

By this I mean splitting the number up into easily manageable groups. This is most easily understood by means of examples.

(*a*) We have just learnt how to multiply by 5. To do this we must multiply half the number by 10, so for 5674×5 we need $5674\div 2$. This is quite a simple exercise if we look at the number in 2-groups

$$56 \mid 74\div 2;\quad \text{the half } 28 \mid 37$$

hits us in the eye, as we say.

Or take 436854×5: as it stands, this is difficult, but in the form

$$4 \mid 36 \mid 8 \mid 54 \times 5$$

the answer $2 \mid 18 \mid 4 \mid 27 \mid 0$ can be written down *with certainty straight away.*

Anyone who wishes to calculate well must learn to look. To start with, split up the number by means of small strokes above, $4'36'8'54$, but later a simple glance will be sufficient. Even without knowing many tricks, group-vision will lighten calculation in many cases.

(*b*) These **examples** have taught us how to halve; *division* by 3, 4, 5, . . . is done in the same way. In *addition* (§ 21) and *subtraction* (§ 26 (1)) the splitting up is nothing but the application of group-vision. There is also a remarkable **application to multiplication,** if one factor is a one-digit number.

3×273 is immediately seen as $3 \times 27'3 = 81'9 = 819$,
6×321 $\qquad 6 \times 32'1 = 192'6 = 1926$.

and with a carry-over from the units

6×323 $\qquad 6 \times 32'3 = 193'8 = 1938$,
8×457 $\qquad 8 \times 45'7 = 365'6 = 3656$.

How troublesome and unpleasant is the method: $(8 \times 400) + (8 \times 50) + (8 \times 7)$! See also § 34(*d*).

(*c*) It is easy to multiply **by 5** in this way:

$$163 \times 5 = 16'3 \times 5 = 8'15 \qquad 1847 \times 5 = 184'7 \times 5 = 92'35.$$

Since $160 \times 5 = 800$, the units product 5×3 is placed as a *two*-digit number after the half. If the tens digit is odd, as in 157×5, the half is $15 \div 2 = 7$, but the units product is now 17×5 and not 7×5; the answer is therefore 785. So we obtain the neat

RULE **To multiply a number by 5, halve the number up to the tens digit and append the units product (as a two-digit number).**

I have chosen this example deliberately to show how, by merely looking at numbers, new relations can arise, and how many methods can lead to the answer. One should remember above all that this makes numbers and calculation come alive. The trickster who looks for a prescription for every exercise like a chemist makes arithmetic too mechanical and so closes new avenues of approach.

(*d*) **When reading out and comparing** large numbers we always split them up into 2-digit groups: not 6,384,053, a number which is difficult to comprehend on first hearing, but 63 84 05 3 . . . nought five, three.

Anyone who is not familiar with 'group-vision' and 'group-diction' will be surprised at the ease and certainty which they provide in calculation, writing down and retention of numbers.

A very good way of exercising the memory is to repeat mentally the numbers of passing cars and to halve or double them.

(*B*) Half and Double of a Two-Digit Number

must be known and not worked out. This greatly increases the flow of a calculation; see above in (*Aa*) the example 5674. No child should work out 56÷2 as 5÷2... etc. Working with *two-* instead of one-digit numbers speeds up the calculation, so that it goes more smoothly and reliably.

Therefore gradually learn the **halves** and **doubles** and (as in § 26 (2)) the **100-complements** of all two-digit numbers, and also the **sum** and **difference** of such numbers.

For teachers of arithmetic. Children often have great difficulty in halving numbers from 1 to 100. Naturally you should first get them to halve only *even* numbers, and to split these up into

'even tens': 20+, 40+, 60+, 80+ and
'odd tens': 30+, 50+, 70+, 90+.

With an even ten the T and U can be halved without remainder, but not with an odd ten. The latter is therefore the case which needs special attention. This slight distinction often works wonders with weak pupils. If you show them that

half of 28 is 14, half of 38 is 19,
half of 46 is 23, half of 56 is 28,

and therefore that *only* the halves of 'odd-ten' numbers can end in 5, 6, 7, 8, 9, you have overcome most of their difficulties.

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§ 30 BY 25

We have

$$\begin{array}{llll} 12\times 25=12\times \frac{100}{4}=\frac{12}{4}\times 100 & & & \\ 13\times 25=(12\times 25)+(1\times 25)=(\frac{12}{4}\times 100)+(1\times 25) & & & \\ 14 & 2 & & 2\times 25 \\ 15 & 3 & & 3\times 25 \end{array}$$

RULE **To multiply a number by 25, multiply a quarter of it by 100** (append two zeros) **and multiply the remainder by 25.**

Examples $28 \times 25 = 7,\ 700$, $464 \times 25 = 116,\ 11600$
35×25: $35 \div 4 = 8$ remainder 3, so $8'75 = 875$
$46 \times 25 = 11'50$, $27 \times 25 = 6'75$, $25 \times 25 = 6'25$
$217 \times 25 = 54'25$, $41683 \times 25 = 1042075$.

Since division by 4 can leave only the remainders 0, 1, 2, 3, multiples of 25 can end only in 00, 25, 50, 75. If these are placed after a quarter of the number, it is *automatically* multiplied by 100.

$$\begin{array}{c} 63593 \times 25 \\ \hline 15898'25 \end{array} \qquad \begin{array}{c} 7284055 \times 25 \\ \hline 1821013'75 \end{array}$$

In two-digit groups this number can easily be quartered (§ 29 (*Aa*)).

Note The 25-trick can naturally be extended to multiplication by 2·5; 250, etc.,

$$\begin{array}{c} 324 \times 2{\cdot}5 \\ \hline 81'0 \end{array} \qquad \begin{array}{c} 324 \times 250 \\ \hline 81'000 \end{array}$$

where the quarter is multiplied by 10; 1000, etc.

Further applications

$$\begin{array}{r} 424 \times 3'25 \\ \hline 10600 \\ 1272 \\ \hline 137800 \end{array} \qquad \begin{array}{r} 346 \times 25'2 \\ \hline 8650 \\ 692 \\ \hline 87192 \end{array} \qquad \begin{array}{l} 32 \times 26 = 8'32 \\ 84 \times 26 = 21'84 \\ 34 \times 24 = 850 - 34 \\ = 816 \end{array}$$

Exercises (i) Multiply by 25: (*a*) 19, 23, 34, 56, 65, 84; (*b*) 93, 123, 146, 268; (*c*) 327, 4584, 97283; (ii) (*d*) 32×625, 4364×525, 48×251, 232×256.

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§ 31 BY 75

We have $4 \times 75 = 300$, that is $75 = \frac{300}{4}$.

$$\begin{array}{lllr} \text{So} & 32 \times 75 = 32 \times \frac{300}{4} = & \frac{32}{4} \times 300 = 8 \times 300 & = 2400 \\ & 33 \quad = & = (\frac{32}{4} \times 300) + (\mathbf{1} \times \ \ 75) & = 2475 \\ & 34 \quad = & + \ \mathbf{2} \times \ \ 75 & = 2550 \\ & 35 \quad = & + \ \mathbf{3} \times \ \ 75 & = 2625. \end{array}$$

RULE **To multiply a number by 75, multiply a quarter of it by 300 and the remainder by 75.**

Since division by 4 can leave only a remainder of 1, 2 or 3 (if it does not divide exactly), only the numbers 75, 150 or 225 will be added to the intermediate answer. So if there is a remainder, multiply a quarter of the number by 3 (instead of 300) and append the end-group as a two-digit number.

Note If the number leaves remainder 3, say 47, take the next higher number, 48, which leaves no remainder and when multiplied by 75 gives a complete hundred, and subtract 75.

Examples Division by 4 (read '3 remainder 1')

$$\begin{array}{r} 13\times 75 \\ \hline 3 \mid 1 \\ \hline 9\quad 75 \end{array} \qquad \begin{array}{r} 47\times 75 \\ \hline 12 \mid -1 \\ \hline 3600 \\ -75 \\ \hline 3525 \end{array} \qquad \begin{array}{r} 135\times 75 \\ \hline 34 \mid -1 \\ \hline 101\quad 25 \end{array} \qquad \begin{array}{r} 4862\times 75 \\ \hline 1215 \mid 2 \\ \hline 3646\quad 50 \end{array}$$

You can very quickly work out in your head these

Exercises Multiply by 75: (*a*) 28, 44, 68, 92, 124; (*b*) 244, 17, 42, 63. (But see also § 51.)

* *

§ 32 BY 125

We have

$$\begin{array}{lll} 16\times 125=16\times \frac{1000}{8}=\frac{16}{8}\times 1000=2000 & & \\ 17 \quad =(16\times 125)+(1\times 125)=2000+125=2125 & & \\ 18 \qquad\qquad\qquad\qquad 2\times 125 \qquad\qquad\qquad\quad 2250, \text{etc.} & & \end{array}$$

RULE **To multiply a number by 125, multiply an eighth of it by 1000 (append three zeros) and the remainder by 125.**

The remainders 1 2 3 4 5 6 7 correspond to the end digits 125 250 375 500 625 750 875, which are easily remembered.

Examples

$$\begin{array}{r} 42\times 125 \\ \hline 5'250 \end{array} \qquad \begin{array}{r} 36024\times 125 \\ \hline 4503'000 \end{array} \qquad \begin{array}{r} 656907\times 125 \\ \hline 82113'375 \end{array}$$

Just as with 25 we have further applications, e.g. $375=3\times 125$, so

$$64\times 375=(64\times 125)\times 3=8000\times 3=24000; \text{ but}$$
$$27\times 375=(27\times 3)\times 125=81\times 125=10'125.$$

For experienced calculators, if the number contains the sequence of digits 125 note the following procedures:

$$\begin{array}{r} 437\times 3'125 \\ \hline 54625 \\ 1311 \\ \hline 1365625 \end{array} \qquad\qquad \begin{array}{r} 322\times 125'4 \\ \hline 40250 \\ 1288 \\ \hline 403788 \end{array}$$

Note For multiplication by 12·5; 1·25; 0·125 the rule naturally holds,

$$\begin{array}{c} 656\times 12{\cdot}5 \\ \hline 82'00 \end{array} \quad \begin{array}{c} 656\times 1{\cdot}25 \\ \hline 82'0 \end{array} \quad \begin{array}{c} 657\times 12{\cdot}5 \\ \hline 82'12{\cdot}5 \end{array}$$

but the eighth is multiplied by 100; 10 and 1 (that is, not at all!)

Exercises Multiply by 125: (*a*) 25, 33, 44, 67; (*b*) 84, 245, 337, 57862.

The numbers 5, 25, 125 belong together as factors of boundaries namely 10, 100, 1000; the method is essentially the same: multiply $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ of the number by the appropriate boundary. On the other hand, each number which we shall now deal with has its *own* rule. Firstly, multiplication

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§ 33 BY 9 AND MULTIPLES OF 9, 18, 27, ETC.

(*a*) Since 9 = 10 − 1; 18 = 20 − 2; 27 = 30 − 3; . . ., the multiplication can be done conveniently by subtraction.

$$\begin{array}{rl} \multicolumn{2}{c}{354\times 27\ (\text{or } 30-3)} \\ \hline 10620 & \times 30 \\ -1062 & \times 3 \\ \hline 9558 & \end{array} \qquad \begin{array}{rl} \multicolumn{2}{c}{8356\times 72\ (\text{or } 80-8)} \\ \hline 668480 & \times 80 \\ -66848 & \times 8 \\ \hline 601632 & \end{array}$$

RULE **Multiply the number by the next complete ten (30, 80) and subtract from this product a tenth of it.**

$$68\times 45=3400-340=3060 \qquad 17\times 2{\cdot}7=51-5{\cdot}1=45{\cdot}9.$$

A neat application: conversion of speed from m/s (read 'metres per second') to km/h (read 'kilometres per hour').

Example An aeroplane flies at the speed of sound, in round figures 340 m/s: how many kilometres does it travel in an hour?

Solution:

$$340\times 60\times 60 \text{ m}=340\times 3600 \text{ m}=340\times 3{\cdot}6 \text{ km}.$$

So we work out

$$(340\times 4)-10\%=1360-136=1224.$$

The aeroplane has a speed of 1224 km/h.

Remember the conversion formula: x m/s = $(4x-10\%)$ km/h.

Exercises What speeds in km/h correspond to

(1) 20, 18, 8, 1·5 m/s; (2) 0·5, 100, 230 m/s?

For the conversion from km/h to m/s the *approximate formula*

$$x \text{ km/h} \simeq (x/4+10\%) \text{ m/s}.$$

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holds. You are travelling at 120 km/h along a motorway. Then your speed is about $30+3=33$ m/s (exactly $33\frac{1}{3}$ m/s).

(*b*) **Special case.** A number whose digits are all equal, such as 77, 888, 3333, can be multiplied by 9 by means of the astonishing

RULE **Multiply one of the equal digits by 9, and put in the middle of the product as many nines as the original number has digits remaining.**

Thus: $77\times 9=693$; $888\times 9=7992$; $3333\times 9=29997$. And $55555\times 9=499995$. You can easily discover the rule if you work out 888×9 at length.

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§ 34 BY 11

(*a*) **Two-digit numbers** can easily be multiplied by 11 mentally:

$11\times 42=462$, $11\times 35=385$, $72\times 11=792$, $11\times 83=913$.

RULE **To multiply a two-digit number by 11, write the sum of the digits between them.**

Example 81×11; split 81 into 8 1 and put in the middle the sum of the digits 9, then 891; say 8, 9, 1.

If the sum of the digits comes to 10 or more, add the ten to the first digit:

$$\begin{array}{r} 67\times 11=6\ 7\\ \underline{13}\\ 737 \end{array}$$

See this in advance, and so increase the T of the two-digit number by 1: $67\times 11=737$. Say 7, 3, 7, or you may easily make the mistake of putting 773.

Exercises (1) Multiply by 11: 24, 36, 72, 66, 97, 58, 82 first semi-mentally, then completely mentally.

(*b*) To multiply **longer numbers** by 11. The answer can be written down straight away. How is this? If we work it out at length we see that each digit is shifted one place to the left, so when they are added this is what happens. The first digit 3 U comes at the end, **3**; the 1 T is added to the 3 U, **4**; the 4 H to the 1 T, **5**; the 5 Th to the 4 H, **9**; finally the **5** Th stands by itself, **5**; so the answer is 59543.

$$\begin{array}{l} \underline{\ 5413\times 11\ }\\ \ \ 5413\\ \underline{5413\ \ }\\ 59543 \end{array}$$

So how do we proceed? If we are asked for 6271×11 we work from right to left: first the end 1; now add 7 and 1, which gives 8; now $2+7=9$ and $6+2=8$ and finally—don't forget—the initial 6: **68981**.

The 6 and 1 form a frame for the answer, as in the case of two-digit numbers, and the sums of each pair of neighbouring digits are placed in the middle. It is like *crochet-work*: we take up the digit next on the left and add it to the one already on the needle. For safety it is best to put a dot over the digits as they are taken up.

So we multiply numbers with several digits by 11 by means of the curious but obvious

RULE OF THUMB **Hook the numbers up!**

Carrying over is naturally taken into account:

$$\begin{array}{l} 8749\times 11 \\ \underline{{\scriptstyle 111}\qquad\quad} \\ \ldots 39 \end{array} \qquad \begin{array}{l} 9;\ \ 4+9=13,\ \ 8+4=12,\ \ 9+7=16,\ \ 1+8=9;\ \text{answer} \\ 96239. \end{array}$$

Exercises (2) Multiply by 11: 4207, 7285, 3928, 67895.

(*c*) **By multiples of 11, such as 22, 33, etc.**

Examples

$$\begin{aligned} 13\times 44 &= (13\times 4)\times 11 = 52\times 11 = 572 \\ 19\times 33 &= \qquad\qquad\quad\ = 57\times 11 = 627 \\ 24\times 33 &= 792 \\ 62\times 77 &= 434\times 11 = 4774. \end{aligned}$$

An experienced calculator can do even the last example in his head. Even young children can do it if they are allowed to give the answer 4774 from right to left.

Exercises (3) 15×55, 17×44, 22×44, 13×77, 33×17, 23×66.

(*d*) **The experienced calculator** notices cases where 11 occurs as part of the multiplier:

$$\begin{array}{rr} & \underline{86\times 115} \\ \times 11 & 946 \\ & \underline{\ \ 430} \\ & 9890 \end{array} \qquad\qquad \begin{array}{rr} & \underline{4864\times 25'11} \\ \times 25 & 121600 \\ \times 11 & \underline{53504} \\ & 12213504 \end{array}$$

He also notices that long numbers can be split up (see § 29 (*A*)) into 2-groups multiplied, often mentally, and the answer written down from left to right;

$$\begin{array}{lllll} \times 11: & \underline{\ 54'13} & \underline{\ 82'32'56} & \underline{36'2} & \underline{3'62} \\ & 594 & 902\ \ 6\,16 & 396 & 33 \\ & \underline{\ \ 1\,43} & \underline{\ \ 3\,52\quad} & \underline{\ \ 2\,2} & \underline{\ 6\,82} \\ & 595\,43 & 905\,58\,16 & 398\,2 & 39\,82. \end{array}$$

Exercises (4) 1397×1125 (see also § 32), 8539×11, 526327×11 (cf. § 37).

§ 35 BY 15

A very fine trick!

RULE **To multiply a number by 15, add half of it and multiply the sum by 10.**

$$24\times15;\quad \begin{array}{r} 24\\ +12\\ \hline 36'0\end{array}\qquad \begin{array}{r}15\times76\\ 38\\ \hline 114'0\end{array}\qquad \begin{array}{r}15\times466\\ 233\\ \hline 699'0\end{array}\quad\cdot\quad \begin{array}{r}15\times8654\\ 4327\\ \hline 12981'0\end{array}$$

For odd numbers add to the number the integral part of half of it, and append 5.

$$\begin{array}{l}17\times15\\ \ \ 8\\ \hline 25'5\end{array}$$

$$\begin{array}{r}15\times427\\ \underline{2135}\\ 6405\end{array}$$

In *written* work it is simpler to work like this: divide the odd digit 7 at the end by 2, giving 3·5, write it without the decimal point, and then nothing needs to be added.

The reader will probably think that 15 should be added, not just 5: but the 1 T is already contained in the odd number 7.

Another way of multiplying by 15 is just as easy for many examples and easier for some:
multiply half the number by 30, that is, multiply it by 3 and append a zero.

$$12\times15=6\times3\mid0=180\qquad 402\times15=201\times3\times10=6030$$
$$54\times15=27\times3\mid0=810\qquad 748\times15=374\times3\times10=11220$$

For *odd* numbers, halve the preceding even number, proceed as before, and add 15:

$$17\times15=(8\times3\mid0)+15=240+15=255$$
$$427\times15=(213\times30)+15=6405$$

Exercises Multiply by 15: (*a*) 22, 46, 64, (*b*) 482, 588, 1754, (*c*) 27, 53, 87, (*d*) 317, 67529. First semi-mentally, then completely mentally. Try both methods.

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§ 36 BY 37

$$3\times37=111;\quad 37=\tfrac{111}{3}.$$

Therefore $$12\times37=12\times\tfrac{111}{3}=\tfrac{12}{3}\times111=444.$$

Note Multiply a third of the number by 111.

So long as a third of the number consists of one digit, as in $27 \times 37 = 999$, there is a significant advantage, but if the number is not divisible by 3 there is less advantage, as in $23 \times 37 = (7 \times 111) + (2 \times 37) = 777 + 74 = 851$ (see also § 37). A similar method applies for 34, 35, 36; a third of the number is multiplied by 102, 105, 108, for example, $45 \times 34 = 15 \times 102 = 1530$.

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§ 37 BY 111

The 111-trick is a neat extension of the 11-trick (§ 34).

(*a*) To multiply a two-digit number by 111:

(1) $53 \times 111 = 5\ 88\ 3$ (2) $67 \times 111 = (6+\mathbf{1})\ \mathbf{4}3\ 7 = 7437$

RULE **Split up the number (53; 67) and put the sum of the digits multiplied by 11 (88; 143) as a two-digit number in the middle.**

The first case can easily be done in the head, and the second after some practice. In this case the sum of the digits is greater than 9; the answer can easily be read out from right to left.

67×111 7; U of $(6+7)$ 3; this increased by 1, $3+1=\mathbf{4}$; T increased by 1, $6+1=7$.

The rule for this method can easily be understood if the example 67×111 is worked out at length.

Application: conversion from degrees of longitude to kilometres. On a great circle 1° corresponds approximately to 111 km, so the shortest flying distance from Hamburg to New York, which differ by 55°, is approximately 6105 km.

Exercises (1) Multiply by 111: 24, 43, 29, 78.

(*b*) To multiply longer numbers with more than two digits by 111.

$$\begin{array}{c}153 \times 111\\ \hline 16983\end{array} \qquad \begin{array}{c}493 \times 111\\ \hline 54723\end{array} \qquad \begin{array}{c}84725 \times 111\\ \hline 9404475\end{array}$$

RULE **Hook the numbers as in the 11-trick (§ 34(*b*)), but take up first one, then two, then three, but never more than three digits on the needle!**

This may seem a very strange rule, but it can easily be understood if it is expressed in this brief pictorial way.

$$\begin{array}{c}153 \times 111\\ \hline 16983\end{array}$$

We reckon from right to left:

3; $5+3=8$; $1+5+3=9$; $1+5=6$; 1.

We naturally carry over if necessary, as in the example above, 493×111. In the example 84725×111 there are three stitches on the needle three times, $(7+2+5)$, $(4+7+2)$, $(8+4+7)$. Naturally at the end there are first two stitches and finally one. How would you multiply by 1111?

Exercises (2) Multiply by 111: 231, 347, 5206, 4835. Calculate 78×37, 14×333.
(3) Multiply by 37 (as in § 36): 216, 327, 1536, 428.

A piece of magic. Ask someone to add together all 6 permutations of a given 3-digit number (say 427). You work out the sum in your head, long before the other person has finished.

The other person works out	You work out
$\begin{array}{r}427\\472\\247\\274\\724\\\underline{742}\\2886\end{array}$	$2\times C(427)\times 111=2\times 13\times 111=26\times 111=2886$ (by *a*). Solution: multiply twice the sum of the digits by 111. Why? In the U, T and H column each of the digits 4, 2, 7 occurs twice. So the answer is $(2C\times 100)+(2C\times 10)+(2C\times 1)=2C\times(100+10+1)$ $=2C\times 111.$

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§ 38 SUMMARY

We must learn how **to multiply by 5, 25, 50, 75, 125, 11 and 15.** The reader must not think, however, that these are exceptional numbers, which occur only in special cases. It is surprising how often the opportunity to use them arises, if one looks out for it.

Apart from this, even if one of these numbers does not occur itself, a number near it may occur, and then we may use our methods to get an exact answer or an estimate.

Exact answers:

34×24; $34\times 25=850$, subtract 34, giving 816.
$422\times 17\ (=15+2)$; $422+211=633$, $6330+844=7174$.

It is often more important in everyday life to obtain an estimate of the answer. If I want to know roughly what 842×26 is, I think of 840×25, and see that the product is therefore more than 21000 and so about 22000. (More about estimates in § 89.)

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The multiplication tables

§ 39 THE LARGE MULTIPLICATION TABLE

From 1 × 1 up to 9 × 9 let us call the 'small'
1 × 11 9 × 19 'medium'
11 × 11 19 × 19 'large' multiplication table.

Many teachers of arithmetic in schools overstress the value of knowing the large multiplication table. We know that everyone must be familiar with the 'small' one in order to do arithmetic. But the large one? No! However much it is drummed into people, they are never as safe with it as they are with the small one. To burden schoolboys with it is a torture, even though it is done with the intention of giving them an important arithmetical tool. The average person will not trust himself with the large multiplication table, because it occurs too seldom.

In any case, the whole effort is *unnecessary*. For there is a surprisingly simple way of working out the product 13 × 18 safely and speedily; as in previous cases this should at first be done partly in one's head, and then it can soon be done entirely mentally. To start with, it is best to write the numbers one under the other:

$$\begin{array}{r}13\\ \underline{18}\\ 210\\ \underline{24}\\ 234\end{array}$$

RULE

(1) Add the U of one number to the other number: 8 + 13 or 3 + 18 = 21;
(2) multiply the sum by 10, that is 210;
(3) add on the U product 3 × 8 = 24: 234.

Examples

$$\begin{array}{rrrrr}14 & 17 & 12 & 19 & 17\\ \underline{18} & \underline{18} & \underline{16} & \underline{16} & \underline{17}\\ 220 & \underline{250} & \underline{180} & \underline{250} & \underline{240}\\ \underline{32} & 306 & 192 & 304 & 289\\ 252 & & & & \end{array}$$

The experienced calculator says only the steps 14 × 18 = 22, 220, 252 and with more practice only the *two* numbers 220, 252.

Everyone who wants to be good at arithmetic must master this wonderful method.

Exercises 16 × 12, 18 × 17, 16 × 16, 19 × 14, 16 × 13.

Note If the U add up to 10, as in 18 × 12 or 13 × 17, you can apply the surprising device of § 49 (1).

The proof may be obtained by those who know algebra from the formula

$$(x+a)(x+b)=(x+a+b)x+ab.$$

For the given example we have

$$(10+3)(10+8)=(10+3+8)\times 10+(3\times 8)=234.$$

We shall see that a whole series of tricks arise from this magic formula (§§ 40, 55, 62 (*A*)).

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§ 40 TWO-DIGIT NUMBERS WITH LIKE TENS,

such as 37 × 34, can be surprisingly worked out by the rule of § 39 with a slight extension.

$$37\times 34\quad \begin{array}{r} 37\\ 34\\ \hline 41\times 3\\ \hline 1230\\ 28\\ \hline 1258 \end{array}\qquad 67\times 62\quad \begin{array}{r} 67\\ 62\\ \hline 69\times 6\\ \hline 4140\\ 14\\ \hline 4154 \end{array}\qquad 74\times 78\quad \begin{array}{r} 74\\ 78\\ \hline 82\times 7\\ \hline 5740\\ 5772 \end{array}$$

RULE **Proceed as in § 39** (add the U of one to the other—append a zero—add the U-product), **but multiply the first sum by the like tens digit.**

Thus $37+4=41,\ \times 3=123,\ 1230,\ +28=1258.$

The proof is again obtained from the simple formula of § 39:

$$37\times 34=(30+7)(30+4)=[(30+7+4)\times 30]+(7\times 4).$$

Exercises 45 × 44, 57 × 58, 73 × 73, 84 × 87. Compare § 46.

Important If the U add up to 10, as in 47 × 43, proceed with the amazing trick of § 49 (1). Compare also § 55.

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§ 41 THE MEDIUM AND SMALL MULTIPLICATION TABLES

(*a*) **The medium table.** How do we work out 8 × 17? Usually like this: $8\times 10=80,\ +56=136$. It is easier just to say 80, 56, **136**. Another example: 9 × 18; say 90, 72, **162**.

Therefore multiply 17 from the *front* by 8 and *say* the answer 'eighty fifty-six'. The addition, giving 136, is surprisingly easy. Say only the answers, the 'steps'.

You should gradually get used to working out *only* the partial answer of the *units*, since that of the T is so easy that it comes automatically. 8 × 17 can then be done in two steps—56, **136**.

The rule can equally well be extended to any case where one number consists of a *single* digit.

6 × 27 = 42, **162**; 84 × 7 = 28, **588**; 9 × 64 = 36, **576** (see also § 29 (*A*,*b*)).

Exercises 23 × 8, 35 × 7, 73 × 8, 47 × 9, 62 × 8, 84 × 4.

For the teacher. The medium table is so important for skilled arithmetic that it should gradually become second nature. Instead of learning the complete 12-, 13-, . . ., 19-tables (1 × 19 up to 9 × 19), as often happens, it is best to start by restricting oneself to products less than 100, and therefore up to 7 × 13 and 5 × 19. In this way one acquires the enthusiasm for working with these important two-digit numbers. It is then not difficult to learn the whole series.

(*b*) **The small table** from 5 × 5 up to 9 × 9 can surprisingly be read off on your *fingers*: 7 × 8; bend down 2 fingers of the left hand and 3 fingers of the right, that is, the numbers by which 7 and 8 exceed 5; multiply the number of fingers bent down (5) by 10: 50; add the product of the numbers of fingers not bent down: 2 × 3 = 6; total 56. So you need only learn the tables from 1 × 1 up to 5 × 5!

In the same way the products from 11 × 11 up to 15 × 15 can be read off on the fingers: 14 × 13; bend down 4 and 3, the excess over 10; their sum is the tens digit: 4 + 3 = 7, so 70; their product 4 × 3 = 12 is the units figure; 70 + 12 = 82 is the excess over 100, so the answer is 182.

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Cross-multiplication

§ 42 THE CROSS METHOD FOR TWO-DIGIT NUMBERS

This method is used by many professional calculators, as it allows the final answer to be written down without any intermediate answers. Naturally it needs practice, but anyone who learns it acquires considerable skill with numbers, and provides himself with facilities which the schoolboy never hears of (e.g. § 47).

Suppose we are given 53 × 46. If we do not resort to any tricks, we must work it out in the old way (*a*).

(*a*)
$$\begin{array}{r} 53\times 46 \\ \hline 318 \\ 212 \\ \hline 2438 \end{array}$$

Look at the answer: it consists of U, T, H, Th. Where do they come from? If we write the numbers one under the other (*b*), we see:

(*b*)
$$\begin{array}{rl} 53 & (1) \\ 46 & (2) \\ \hline 2438 & (3) \end{array} \qquad \begin{array}{ll} U_3=U_1\times U_2=3\times 6= & 18 \\ T_3=(U_1\times T_2)+(U_2\times T_1)=(3\times 4)+(6\times 5)= & 42 \\ H_3=T_1\times T_2=5\times 4= & \underline{20} \\ & 2438 \end{array}$$

Since the answer is obtained in this way, it can be worked out in one go like this.

$$\begin{array}{r} 5\;\;3 \\ |\times| \\ 4\;\;6 \\ \hline 2\,4\,3\,8 \end{array}$$

3 × 6 = 18; carry 1 T;
5 × 6 = 30, + (3 × 4) = 42, + 1 = **43**, carry 4 H;
5 × 4 = 20, + 4 = **24**, so 2438.

This is a great increase in speed, since the partial answers are left out. More important still is the insight into the decisive steps which lead to the answer. Previously we have simply worked as in (*a*): multiply first by 6, then by 4, shifting one place, then add. We have not noticed which numbers are important for the answer. Which are important? All of them, of course! But obviously the error is not so great if we get **2434** rather than **2238** or **3438**. Previously we just worked straight through, but with cross-multiplication we work out *each unit* by itself and know exactly where we must pay particular attention.

And where is this? It is in the tens, in the **cross**. This is the heart of the answer.

So we work out the cross first.

$$\begin{array}{r} 56 \\ 23 \\ \hline 1288 \\ \scriptstyle 2 \\ bac \end{array}$$

We work out 56 × 23 like this. The reader should say it to himself.
(*a*) Cross: 15 + 12 = 27, + 1 = **28** (1 is the U-carry-over)
(*b*) H: 10, **12**
(*c*) U: **8** (The 1 T from 18 has already been counted under (*a*)).

We will go through **the individual steps** once more.

(*a*) I add the U-carry-over *immediately* to the cross. I do not need to work out the U first, but I *look at* the carry-over. The reader should under no circumstances say the U-product 3 × 6; he will very soon notice how quickly the brain delivers the carry-over, more quickly than if one had asked it politely: please work out 6 × 3, but I shall only use the

T! Frequently in arithmetic the fault lies not in lack of practice but in lack of confidence in one's own ability!

So look at the carry-over and count it in with the cross: 27, **28.** The 20 T gives the H-carry-over 2; we write this to our credit small and below, one place to the left.

(*b*) Now we *see* the H-answer (5 × 2 =) 10 (don't read out the sum, it's just the small multiplication table) and add on the H-carry-over: 10, **12.**

(*c*) Finally we put in the U: 8.

So we multiply the numbers according to the scheme |×|, a 'cross' between two 'posts', and remember the

RULE OF THUMB **First the cross and then the posts!** (T-post first!)

Fix these steps *a*, *b*, *c* in your mind, if you want to master this useful method of multiplication. Don't think that it is the same if you begin with the U and not with the cross. Anyone with a feeling for numbers knows that it is not the same.

The sign × arises from the 'cross' of cross-multiplication.

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§43 TWO HINTS FOR CROSS-MULTIPLICATION

should gradually be learnt:

(1) First test *with which arm* of the cross it is best to *begin*; generally it is easier to add the smaller number to the larger, so begin with the larger:

87 49 + 16, 87 40 + 63, or just as simply 63 + 40.
/ not 16 + 49. \
$\underline{72}$ $\underline{95}$

It is probably better to take complete T first.

(2) Gradually get into the habit of adding the T (16 to 49 in the first example) without saying them, but just giving the steps: 49, **65.**

94	(*a*) 45, 61, 63	76	56, 80, **84**
$\underline{45}$	(*b*) 36, 42	$\underline{48}$	28, 36
4230	(*c*) 0	3648	8
6		8	

Exercises Write one under the other and cross-multiply:

(*a*) 41 × 31, 68 × 41, 24 × 18, 45 × 37.

(*b*) 57 × 33, 27 × 68, 83 × 86, 52 × 39.

(*c*) 96 × 89, 47 × 105, 114 × 32.

(*d*) 123 × 48, 156 × 44, 253 × 82, 63 × 8.

Make up further exercises for yourself, until you become fluent with the method. From now on, multiply **two-digit numbers** by the cross, even if it serves no other purpose; if, to start with, you cannot forget the old method, work with that, but test with the new. You will soon realise its great superiority: it is much quicker, safer and more attractive. Cross-multiplication has particular value because of a series of *simplifications*, the most surprising of which you will find in §§ 46 and 47.

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§ 44 LARGER NUMBERS

can be dealt with by the experienced calculator by the methods already learned, if he regards the H as T:

$$\begin{array}{rrrrr} 17|8 & 135 & 154 & 256 & 406 \\ 4|2 & 144 & 221 & 142 & 82 \\ \hline 7476 & 19440 & 34034 & 36352 & 33292 \end{array}$$

These illustrate respectively: the medium multiplication table, the large table, multiplication by 15, by 25, and complete tens.

Exercises (*a*) 186 × 45, 167 × 185, 155 × 421; (*b*) 242 × 254, 87 × 605.

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§ 45 CROSS-MULTIPLICATION FOR NUMBERS WITH MORE DIGITS

is unfortunately not so simple as for two-digit numbers and can only be done by clever calculators:

328	U gives the U-post 8×6		
576	T	TU-cross $(2 \times 6) + (7 \times 8)$	48
	H	T-post + HU-cross $(2 \times 7) + [(3 \times 6) + (5 \times 8)]$	68
	Th	TH-cross $(3 \times 7) + (5 \times 2)$	72
	T Th	H-post 3×5	31
			15
			188928

To do all this, carrying over mentally, requires a good memory for figures, but there are people who can do it without much trouble. What is there to do? Just what appears in symbol form:

$$\begin{array}{cccccc} & | & \times & \ast & \times & | \\ \text{T Th} & \text{Th} & \text{H} & \text{T} & \text{U} & \end{array}$$

which is remembered as Post—Cross—Star—Cross—Post!

Remark for the experienced calculator: 398 × 746 gives several high partial products, for example, in the H-star: 36 + 18 + 56. This can be avoided by introducing the idea of *negative* 'digits', and so writing 398 = 400 − 2 as $40\bar{2}$, in which $\bar{2}$ counts as −2. Then we have

$$\begin{array}{r l} 40\bar{2} & -12;\quad -8-1=-9;\quad 24-14=10 \\ \underline{746} & 16,\ 17;\quad 28,\ 29 \\ 2970\underline{92} & \text{Answer } 297000-92=296908. \end{array}$$

In this attractive way the clever calculator who knows how to deal with negative numbers can save himself considerable work; he need only work through the example with 398 to see this; there is a still easier method in § 56 (1).

The difficulty of the method lies in the cross, and it is increased if the numbers are still bigger. Can we not do away with the folding of the numbers in the cross? Yes, if we *reverse* the upper number and *slide* it over the lower according to the so-called

(*a*) **Sliding method.** An example with two-digit numbers. 48 × 76 is reversed to give 84 × 76; now slide the 84 from right to left one place at a time over the 76 and always multiply in post or *columns*:

$$\begin{array}{ccc} (a) & (b) & (c) \\ 84 & 84 & 84 \\ 76 & 76 & 76 \end{array}\qquad \begin{array}{l r} \text{U: } 8\times 6= & 48 \\ \text{T: } (4\times 6)+(8\times 7)= & 80 \\ \text{H: } 4\times 7= & \underline{28} \\ & 3648 \end{array}$$

In this way the T-cross (*b*) is changed into two *columns*. Here lies the advantage of this method, which we shall carry through for 328 × 576.

RULE (1) **Reverse the sliding number (328 → 823. Look out for a slip of the pen!)**

(2) **Slide the number (823) from right to left one place at a time (over the 576), add up the column products mentally and write down one digit; carry over to the next stage.**

$$\begin{array}[t]{r} \text{U} \\ 823 \\ \underline{576} \\ 48 \end{array}\qquad \begin{array}[t]{r} \text{T} \\ 823 \\ \underline{576} \\ 48 \\ 12 \\ \underline{56} \\ (728) \end{array}\qquad \begin{array}[t]{r} \text{H} \\ 823 \\ \underline{576} \\ 728 \\ 18 \\ 14 \\ \underline{40} \\ (7928) \end{array}\qquad \begin{array}[t]{r} \text{Th} \\ 823 \\ \underline{576} \\ 7928 \\ 21 \\ \underline{10} \\ (38928) \end{array}\qquad \begin{array}[t]{r} \text{T Th} \\ 823 \\ \underline{576} \\ 38928 \\ \underline{15} \\ 188928 \end{array}\qquad \begin{array}[t]{r} (3) \\ \\ \\ \\ \underline{576} \\ 188928 \\ {\scriptstyle 3774} \end{array}$$

All that is actually written is the working under (3).

Exercises (see below) This method can be valuable if one has to multiply a series of numbers by the same sliding number, as for example in working out tables.

(*b*) **The funnel method** is much simpler, as it deals with *one* cross at a time. The reader can learn the idea best from § 60, which I would ask him to read first. Then he will understand the example 328 × 576:

$$\begin{array}{r}
3\;2\;8 \\
\underline{5\;7\;6} \\
151448 \\
3168 \\
\underline{58} \\
188928
\end{array}$$

Instead of squares, deal with columns, instead of steps, deal with crosses:

(1) Columns $3\times5=15$, $2\times7=14$, $8\times6=48$

(2) Cross $(3\times7)+(5\times2)=31$, $(2\times6)+(7\times8)=68$

(3) 1-cross $(3\times6)+(5\times8)=58$.

In a 1-cross we jump over 1 digit, in a 2-cross we jump over 2-digits, and so on. If a cross-product exceeds 100, proceed as in § 60 (3). This attractive method has a useful

Application: abbreviated multiplication of decimals.

Suppose $6{\cdot}3457\times5{\cdot}6728$ is required to 2 places of decimals. The answer contains 8 places, of which 6 are of no use. We proceed like this:

$$\begin{array}{r|l}
6{\cdot}34 & 57 \\
5{\cdot}67 & 28 \\
\hline
30{\cdot}18 & 3 \\
4\;14 & 5 \\
62 & 4 \\
3 & 8 \\
 & 8 \\
\hline
35{\cdot}99 & 8 \\
\simeq 36{\cdot}00 &
\end{array}$$

After the second place we draw a vertical line and work out the funnel from the left up to the third place after the decimal point, rounding off if necessary; thus $6\times5=30$, $3\times6=18$, $4\times7\simeq3|0$. All considerations of the order of magnitude, which normally hold up the work, are eliminated in this way. The exact answer is 35·99788696.

Cross $51+45$.

1-cross $42+20=62$; $6+30\simeq4|0$.

2-cross $12+25=37$; $24+42\simeq7|0$; 37·7 rounded off to 38.

3-cross $48+35\simeq8|0$.

Note (1) In order to form the funnel properly, write each product as a **2**-digit number, so $8\times0=00$ (example (*b*), below) and $0\times0=00$ (example (*a*)).

(2) Always begin with the U-column (the one *before* the decimal point, in heavy type in the examples); $5\times3=\mathbf{15}$, so the 5 U goes under the U of the factors (example (*c*)) and $5\times0=\mathbf{00}$ (example (*d*)). All other column products (in the first line) and all cross-products then go in their right places.

(3) The order of magnitude of the answer can be found by estimation ($85\times3\simeq250$, example (*c*)).

Examples All to 3 places: 8534 × 3127 = 26685818.

(*a*)
$$\begin{array}{r|l} 0{\cdot}853 & 4 \\ 0{\cdot}312 & 7 \\ \hline 00\ 240 & 5 \\ 0\ 023 & 1 \\ 002 & 5 \\ 00 & 7 \\ \hline 0{\cdot}267 & \end{array}$$

(*b*)
$$\begin{array}{r|l} 8{\cdot}534 & \\ 0{\cdot}312 & 7 \\ \hline 00\ 150 & 3 \\ 2\ 414 & 1 \\ 082 & 2 \\ 16 & 4 \\ 5 & 6 \\ \hline 2{\cdot}669 & \end{array}$$

(*c*)
$$\begin{array}{r|l} 85{\cdot}340 & \\ 3{\cdot}127 & \\ \hline 0015\ 030 & 8 \\ 241\ 410 & 3 \\ 08\ 222 & 2 \\ 1\ 635 & \\ 56 & \\ \hline 266{\cdot}858 & \end{array}$$

(*d*)
$$\begin{array}{r|l} 85{\cdot}340 & 0 \\ 0{\cdot}312 & 7 \\ \hline 0000\ 090 & 4 \\ 001\ 515 & 1 \\ 24\ 050 & 6 \\ 0\ 810 & 2 \\ 163 & 5 \\ 56 & \\ \hline 26{\cdot}686 & \end{array}$$

In example (*a*), if both decimals begin with 0, . . ., you can start with the first place of decimals (8 × 3 = 24). The 00-group at the front of each row then drops off. Watch out where the 24 goes!

Exercises (*a*) 374 × 384 627 × 36 (*b*) 734 × 512 614 × 342
(*c*) 8307 × 2312 (*d*) 3·1415 × 8·7362 to 1, 2, 3 places.

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§46 THE TRICK OF THE DOUBLE DIGIT

is an attractive special case of cross-multiplication. What is a double digit (DD)?

$$(a)\ \begin{array}{r}33\\ \underline{52}\end{array} \qquad (b)\ \begin{array}{r}61\\ \underline{44}\end{array} \qquad (c)\ \begin{array}{r}52\\ \underline{57}\end{array} \qquad (d)\ \begin{array}{r}34\\ \underline{74}\end{array} \qquad (e)\ \begin{array}{r}37\\ \underline{43}\end{array} \qquad (f)\ \begin{array}{r}45\\ \underline{52}\end{array}$$

(*a*) to (*d*) are the four possible cases of the double digit: T and U are equal (*a*, *b*); T are equal (*c*); U are equal (*d*).

Note **The double digits** are **across** or **down**, not oblique as in (*e*) and (*f*).

What does the trick consist of? We see from case (*a*). The cross is (2 × 3) + (5 × 3), which is (2 + 5) × 3 = 7 × 3. So we can work out the cross very easily: we add the remaining digits—those which are not the DD—and multiply the sum by the DD. We remember the

RULE OF THUMB **for the cross: DD times the sum of the others!**

The simplification applies only to working out the cross; the rest goes as before.

Examples Case (*b*) (1 + 6 =)7, × 4 = 28; 24, 26; 4; so 2684.

The cross in cases (*c*) $5\times9=45$; (*d*) $4\times10=40$.

$$\begin{array}{rrrrrr}
76 & 66,69 & 88 & 72,75 & 54 & 4\times18=72,73\\
46 & 28,34 & 54 & 40,47 & 134 & 65,72\\
\cline{1-1}\cline{3-3}\cline{5-5}
 & 6 & & 2 & & 6\\
3496 & & 4752 & & 7236 & \\
{\scriptstyle 6}\quad\; & & {\scriptstyle 7}\;\; & & {\scriptstyle 7}\;\; &
\end{array}$$

What could be easier?

Exercises

$$\begin{array}{llllllllll}
(a) & 32 & 45 & 33 & 45 & 87 & (b) & 93 & 93 & 88\\
 & \underline{42} & \underline{43} & \underline{26} & \underline{77} & \underline{85} & & \underline{22} & \underline{83} & \underline{72}\\
 & 95 & 72 & 103 & 127 & & (c) & 21 & 61 & 45 & 135\\
 & \underline{95} & \underline{27} & \underline{108} & \underline{122} & & & \underline{41} & \underline{51} & \underline{65} & \underline{65}
\end{array}$$

Special cases (in Exercise *c*)

(1) **Product of 1-numbers** (numbers ending in 1). If the DD is 1 and is the U in each factor, as in 31×41 or 71×51, the answer can be written down immediately.

$$\begin{array}{lll}
H=t_1\times t_2 & T=t_1+t_2 & U=1\\
\;\;=3\times4 & \;\;=3+4 & \;\;=1;\ 41+31=1271;\ 12,\ 7,\ 1.\\
\;\;=7\times5(+1) & \;\;=7+5=(1)2 & \;\;=1;\ 71+51=3621\ \ \text{(T-carry-over)}
\end{array}$$

RULE Write the **T-sum** as a *one*-digit number between the **T-product** and **1**.

(2) **Product of 5-numbers** such as (*a*) 85×65 and (*b*) 55×85 and (*c*) 35×45.

(*a*) The T-sum $8+6=14$ is *even*:

$85\times65=(48+7)\mid25=5525$; $115\times75=(77+9)\mid25=8625$.

RULE **T-product × half T-sum, then 25.**

(*b*) The T-sum $5+8=13$ is *odd*:

$55\times85=(40+6)\mid75=4675$; $35\times65=(18+4)\mid75=2275$.

RULE **T-product + integral part of half the T-sum, then 75.**

(*c*) The 5-numbers are *neighbours*. So

$35\times45=3\times(3+2)\mid75=3\times5\mid75=1575$;

$85\times95=8\times10\mid75=8075$; $15\times25=1\times3\mid75=375$.

RULE **Smaller T times this increased by 2, then 75.**

For the mathematician, here is the proof of this astonishing rule ($t=$ ten digit):

$$\begin{aligned}
(10t+5)[10(t+1)+5] &= 100t(t+1)+5[10t+10(t+1)]+25\\
&= 100t(t+1)+5.2t.10+50+25\\
&= 100t(t+1)+100t+75=100t(t+2)+75.
\end{aligned}$$

Hint. Case (*c*) can be done as in § 54. The special case of 5-numbers with equal tens, such as 35 × 35, as in § 62(*A*).

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§ 47 DOUBLE DIGIT WITH 10-COMPLEMENTS,

a further special case which we meet in cross-multiplication, is a more genuine 'trick'.

If the remaining digits add up to 10, as in

$$(a)\ \begin{array}{r}43\\ \underline{47}\end{array}\qquad (b)\ \begin{array}{r}22\\ \underline{64}\end{array}\qquad (c)\ \begin{array}{r}27\\ \underline{87}\end{array}\qquad (d)\ \begin{array}{r}73\\ \underline{66}\end{array}$$

we can immediately write down the answer from left to right by means of the surprising

RULE OF THUMB **T-post + DD, then U-post!**

Therefore: (*a*) (4 × 4) + 4 = 20; 20 | 21 (*c*) 2349
(*b*) 12 + 2 = 14; 1408 (*d*) 4818.

The proof is easily obtained by working through as in § 46. The cross is always a complete **T**, and so is that of the DD, (in (*a*) 4 × 10 = 40), so that the U (21) can be written in full, as the carry-over digit 2 takes the place of the 0 in 40. The carry-over from the cross to the T-post is always the DD.

So if a DD occurs in an exercise a quick glance at the remaining digits will show whether they add up to 10. If they do, you are abundantly rewarded; if they do not, proceed as in § 46.

Examples

$$\begin{array}{r}48\\ \underline{68}\\ 3264\end{array}\quad \begin{array}{l}\text{T-product: } 4\times 6=24+\text{DD } 8=32;\\ \text{U-product: } 8\times 8=64, \text{ so } 32|64.\\ \end{array}\qquad \begin{array}{r}33\\ \underline{46}\\ 1518\end{array}\quad \begin{array}{r}12,\ \mathbf{15}\\ \mathbf{18}\\ \end{array}$$

Warning. For simple 10-complements *without* DD our magic formula breaks down, as for example 47 × 53.

Exercises

$$(a)\ \begin{array}{r}22\\ \underline{28}\end{array}\ \ \begin{array}{r}64\\ \underline{33}\end{array}\ \ \begin{array}{r}45\\ \underline{45}\end{array}\ \ \begin{array}{r}87\\ \underline{27}\end{array}\ \ \begin{array}{r}41\\ \underline{49}\end{array}\ \ \begin{array}{r}47\\ \underline{88}\end{array}\ \ \begin{array}{r}37\\ \underline{55}\end{array}$$

$$(b)\ \begin{array}{r}66\\ \underline{19}\end{array}\ \ \begin{array}{r}91\\ \underline{33}\end{array}\ \ \begin{array}{r}104\\ \underline{106}\end{array}\ \ \begin{array}{r}147\\ \underline{143}\end{array}\qquad (c)\ \begin{array}{r}246\\ \underline{244}\end{array}\ \ \begin{array}{r}112\\ \underline{118}\end{array}\ \ \begin{array}{r}163\\ \underline{167}\end{array}\ \ \begin{array}{r}308\\ \underline{302}\end{array}$$

The experienced calculator can look into the cases of **DD with complements to multiples of 10,** where the remaining digits add up to 20 = 2 × 10, or 70 = 7 × 10, or, generally, to **n** × 10.

Examples **187** × **27**, **264** × 22, **863** × **43**. He will easily discover the

RULE **T-post + n times the DD, then U-post!**

So $187 \times 27 = [(18 \times 2) + (2 \times 7)] \mid 49 = (36 + 14) \mid 49 = 5049$; then $[(26 \times 2) + (3 \times 2)] \mid 08 = 5808$ and $[(86 \times 4) + (9 \times 3)] \mid 09 = (344 + 27) \mid 09 = 37109$. (Write the factors one under the other.)

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§ 48 SUMMARY

Cross-multiplication is particularly attractive because of the insight it gives into the texture of a product, and is useful because of a series of surprising special cases. The reader should master particularly the multiplication of two-digit numbers in §§ 42 and 43, the trick of the double-digit in § 46, and its wonderful special case of 10-complements in § 47. In the course of dealing with this he will discover further refinements, such as the one where the cross can be written out completely, when a number in the fifties is multiplied by a number with an even ten, such as 84 × 53. I leave to the reader the task of developing the method for numbers with more than three digits.

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Column multiplication

§ 49

Why does the case of the double digit in § 47 give us so much pleasure? Chiefly because we do not need to work out the 'cross', but simply the 'columns', and therefore multiply T by T and U by U. What a pity we cannot always work like that! But we can. We shall now deal with a most beautiful and surprising trick. We shall separate our exercises into those with like tens (63 × 68) and those with unlike tens (63 × 48), in order to proceed to the main rule. The case of like tens is particularly valuable; the other case can be dealt with only by experienced calculators.

(1) T-like numbers with 10-complements

If the DD is the T, the numbers are T-like, and if the remaining digits add up to 10, as in 63 × 67, we can apply this wonderful

RULE **Multiply the T by the T increased by 1, and then append the U-product!**

$63 \times 67 = 6 \times 7 \mid 21 = 4221$ $16 \times 14 = 2 \mid 24$ (large multiplication table)
$42 \times 48 = 4 \times 5 \mid 16 = 2016$ $153 \times 157 = 15 \times 16 \mid 21 = 24021$.

Every calculator must have this trick up his sleeve! It is, as one can easily show, an application of § 47.

Exercises (1) of which the answers can be written down immediately.

$$\begin{array}{cccccc} 16 & 45 & 59 & 92 & 163 & 243 \\ \underline{14} & \underline{45} & \underline{51} & \underline{98} & \underline{167} & \underline{247} \end{array}$$

Use this device also in *combinations*:

482×42 (48×42); 636×34.

You will see how worth-while it is to look at the numbers carefully.

(2) T-like numbers without 10-complements

$$\begin{array}{rrrlrlrrl} 67 & (a) & \overset{1}{67} & & \overset{2}{67} & & (b) & \overset{-2}{67} & \\ \underline{\times 63} & & \underline{64} & & \underline{65} & & & \underline{61} & \\ 42|21 & & 4228 & & 4235 & & & 4207 & \\ \text{(Case 1 above)} & & \underline{+6} & +1\times\mathrm{DD} & \underline{+12} & +2\times\mathrm{DD} & & \underline{-12} & -2\times\mathrm{DD} \\ & & 4288 & & 4355 & & & 4087 & \end{array}$$

The U-sum exceeds 10 in case (*a*) and falls short in case (*b*). The difference from 10 is written small at the top. This gives the beautiful *RULE* **Multiply in columns** (the T by the T increased by 1) **and add or subtract the product of the 10-difference and the DD** (+2 × DD; −2 × DD) **as a ten** (and therefore in the second place).

The proof is easily found by going through the exercises using the method of (1) above.

Exercises (2) (*a*) Apply this rule to the large multiplication table from 13 × 13 onwards: 14 × 14, 17 × 18, and so on.

$$\begin{array}{lcccccc} (b) & 47 & 23 & 56 & 87 & 123 & 256 \\ & \underline{48} & \underline{29} & \underline{53} & \underline{84} & \underline{129} & \underline{258} \end{array}$$

(3) T-unlike numbers with and without 10-complements. The experienced calculator can develop these cases himself. Apart from many attractive discoveries, he will learn that in the general case column multiplication leads back to a cross-multiplication.

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Multiplication of numbers near a boundary

§ 50 THREE CASES

Up to now we have dealt with the main methods for multiplication. We must be able to multiply a *given pair of numbers*, and *cross-* and *column-multiplication* provide us with excellent tools at any rate for numbers with two digits. Now we are going to deal with certain special cases, such as numbers *near a boundary*, the so-called *squares*, and so on, which we can often work out surprisingly simply. We now deal with the first.

By numbers *near a boundary* we understand those numbers which lie near a **unit-boundary**, such as 100, 1000, 10000.... (See also § 56 (1).)

Three cases are possible:

(*a*) 98 × 94, where both numbers lie **below** the boundary (100): § 51;
(*b*) 102 × 105, where both numbers lie **above** the boundary (100): § 52;
(*c*) 94 × 107, which **enclose** the boundary (100): §§ 53 and 54.

In all these cases the answer can be found surprisingly simply, if one uses the *distance* from the boundary. The distance of the number *below* 100 (1000, ...) is the 100- (1000-, ...) *complement*.

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§ 51 THE NUMBERS LIE BELOW THE BOUNDARY, 98 × 94

RULE (1) **Determine the 100-complements** (98 + **2** = 100 and 94 + **6** = 100) **and write them beside the numbers.**

(2) **Subtract from the first number, 98, the 100-complement 6 of the first** (or subtract from the second, 94, the 100-complement 2 of the first, whichever is easier): **92.**

98 2
94 6
9212

(3) **Append the product of the complements** 2 × 6 = 12: **9212.**

Examples

95 5
89 11
84 55

98 2
97 3
9506

75 25
82 18
57
4 50
61 50

993 7
995 5
988035

9985 15
9992 8
9977|0120

983 2
96 4
9408
+(3 × 96) 288
94368

Place rule. Write the product of the complements with as many **places** as the boundary has **zeros.**

Calculations like these can actually be done in one's head. Look at the application to 983 × 96.

According to his skill a calculator can go further and further from the boundary, provided that he can easily work out the product of the complements, either by a trick or as here by cross-multiplication. This is an excellent training for quick and enjoyable calculation, which no teacher should neglect.

9963 37
9942 58
99052146

The clever calculator can naturally do things like this

$$0{\cdot}95\times0{\cdot}93=0{\cdot}8835;\quad 9{\cdot}2\times0{\cdot}88=8{\cdot}096$$

Proof of the method. We choose our example $98\times94=a\times b$.

$$\begin{array}{ll} 98=a & \text{100-complement: } 100-a=2 \\ 94=b & \qquad\qquad\qquad\quad\; 100-b=6 \end{array}$$

We now write out the method in terms of the general numbers a and b.

$$\begin{aligned} &100[a-(100-b)]+(100-a)(100-b) \\ &=100a-100^2+100b+100^2-100a-100b+ab=ab. \end{aligned}$$

In this remarkable method all the terms cancel except ab.

Exercises (*a*)

$$\begin{array}{cccccc} 97 & 88 & 95 & 98 & 79 & 78 \\ \underline{92} & \underline{93} & \underline{95} & \underline{82} & \underline{97} & \underline{92} \end{array}$$

(*b*)

$$\begin{array}{ccccccc} 994 & 992 & 980 & 9995 & 9972 & 9942 & 982 \\ \underline{996} & \underline{984} & \underline{978} & \underline{9980} & \underline{9985} & \underline{9975} & \underline{94} \end{array}$$

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§ 52 NUMBERS ABOVE THE BOUNDARY

What happens if the numbers in the neighbourhood of a boundary *exceed* it? There are two ways:

(*a*) As in § 51, but using *excess* instead of complement.

RULE (1) Add the 100-excess of one number to the other number: 115.

(2) Write after it the excess-product 50; as a 2-digit number if the boundary is 100, as a 3-digit if the boundary is 1000, and so on.

$$\begin{array}{rr} 105 & 5 \\ \underline{110} & \underline{10} \\ 115| & 50 \end{array}$$

Examples

$$\begin{array}{rr} 1020 & 20 \\ \underline{1015} & \underline{15} \\ 1035| & 300 \end{array}\qquad \begin{array}{rr} 10052 & 52 \\ \underline{10011} & \underline{11} \\ 10063| & 0572 \end{array}\qquad \begin{array}{rr} 125 & 25 \\ \underline{118} & \underline{18} \\ 143 & \\ \underline{\;\;450} & \\ 14750 & \end{array}$$

(*b*) **More simply,** however, according to this

RULE **Add the two numbers together, subtract the boundary, and append the excess-product.**

That is, not **215|50** but **115|50** in the above example. The reader can easily provide the proof of this surprising trick as in § 51; rule (*b*) can also be applied for numbers below a boundary, but I prefer the method of § 51.

Exercises

$$\begin{array}{cccccc} 107 & 108 & 115 & 125 & 1010 & 10011 \\ \underline{109} & \underline{114} & \underline{134} & \underline{148} & \underline{1037} & \underline{10063} \end{array}$$

* * *

§ 53 MULTIPLICATION OF NUMBERS WHICH ENCLOSE THE BOUNDARY,

that is, where one lies below and the other above.

97 × 106 97 is 3 below 100, −3
106 is 6 above 100, +6.

$$\begin{array}{rr} 97 & -3 \\ 106 & +6 \\ \hline 10300 & \\ -18 & \\ \hline 10282 & \end{array}$$

RULE (1) **Add the excess of one number to the other or subtract the complement of one from the other: 103.**
(2) **Append as many zeros as the boundary has: 10300.**
(3) **Subtract the product of the excess and the complement: −18, giving 10282.**

Examples

$$\begin{array}{rr} 86 & -14 \\ 115 & +15 \\ \hline 10100 & \\ -210 & \\ \hline 9890 & \end{array} \qquad \begin{array}{rr} 84 & -16 \\ 103 & +3 \\ \hline 8700 & \\ -48 & \\ \hline 8652 & \end{array} \qquad \begin{array}{rr} 997 & -3 \\ 1008 & +8 \\ \hline 1005000 & \\ -24 & \\ \hline 1004976 & \end{array}$$

The proof is as before (§ 51).

Exercises

$$\begin{array}{cccccc} 94 & 88 & 78 & 92 & 85 & 996 \\ \underline{106} & \underline{109} & \underline{107} & \underline{115} & \underline{144} & \underline{1012} \end{array}$$

* * *

§ 54 SPECIAL CASE: MIRROR NUMBERS

If two numbers, such as 47 and 53, enclose a boundary, which need not be a unit-boundary, so that they are at *equal distances* from it, 47–50–53, and therefore the complement (3) and the excess (3) are equal, we call them *mirror-numbers*; the mirror is 50. Then the method of § 55 applies, which we now anticipate:

$$\begin{array}{rr} 47 & -3 \\ 53 & +3 \\ \hline \end{array}$$

$$5 \times 50 \times 10 - 9 = (50 \times 50) - (3 \times 3) = 50^2 - 3^2,$$

that is, multiply the '*mirror*' by *itself* (50 × 50), and subtract the *distance* multiplied by itself (3 × 3). Answer 2491

Those who know some algebra will recognise the formula $(a+b)(a-b)=a^2-b^2$; here a is the mirror, b the distance. Using the idea of 'squares', we have the simple

RULE **The product of two mirror-numbers is equal to the difference between the squares of the mirror and the distance.**

Examples $87 \times 93 = 90^2 - 3^2 = (90 \times 90) - (3 \times 3) = 8100 - 9 = 8091$.
$152 \times 148 = 150^2 - 2^2 = 22500 - 4 = 22496$.

Exercises 69×71, 46×54, 25×35, 94×106, 248×252, 874×866.

Since this method is only applied when the mirror is a T-boundary and the U add up to 10, the experienced calculator can also work as in § 49 (2*a*).

The product of 5-numbers (those ending in 5) can easily be worked out by the above rule:

(*a*) Neighbours, such as 55×65; the mirror is 60, so
$55 \times 65 = 60^2 - 5^2 = 3600 - 25 = 3575$.
$85 \times 95 = 90^2 - 25 = 8075$; $115 \times 125 = 14400 - 25 = 14375$.

A very fine trick!

(*b*) **Non-neighbours,** such as 35×65 and 45×65
35×65 mirror $(35+65)/2 = 50$; $50^2 - 15^2 = 2500 - 225 = 2275$.
45×65 mirror $110/2 = 55$, so
$45 \times 65 = 55^2 - 100 = 3025 - 100 = 2925$.
$35 \times 75 = 55^2 - 400 = 2625$.

The mirror, which is always the mean of the two factors, here gives either a complete T or a 5-number, whose square can easily be formed as in § 62 (*A*). Compare also § 46 (2).

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§ 55 IF THE BOUNDARY IS NOT A UNIT-BOUNDARY,

the methods of §§ 51 to 53 can still be applied, for example 57×56 (upper boundary 60, lower 50), or 895×893 (boundary 900), or 705×712 (boundary 700), **if one inserts a multiplication by the boundary digit** (6 for 60, 600, 6000, and so on). So for the example 57×56:

(*a*) upper boundary 60	(*b*) lower boundary 50
(as in § 51)	(as in § 53)
57 3	57 7
56 4	56 6
53×6	63×5
318	315
12	42
3192	3192

RULE (1) **If the upper boundary (60) is chosen subtract the distance from the boundary from the other number: $57-4=53$ (add in the case of lower boundary (50): $57+6=63$);**

(2) **multiply the resulting number by the boundary digit: $53\times 6=318$ (or $63\times 5=315$);**

(3) **append the product of the distances $3\times 4=12$ (or $7\times 6=42$), written as a one-digit (!) number: 3192.**

In (1) for the upper boundary the distance is negative, $57=60-3$, for the lower it is positive, $57=50+7$.

Examples

$$\begin{array}{r} 895\ \ 5 \\ 893\ \ 7 \\ \hline 888\times 9 \\ \hline 7992\mid 35 \end{array} \qquad \begin{array}{r} 3992\ \ 8 \\ 3985\ \ 15 \\ \hline 3977\times 4 \\ \hline 15908\mid 120 \end{array} \qquad \begin{array}{r} 705\ \ 5 \\ 712\ \ 12 \\ \hline 717\times 7 \\ \hline 5019\mid 60 \end{array} \qquad \begin{array}{r} 705\ \ +5 \\ 694\ \ -6 \\ \hline 699\times 7 \\ \hline 489300-30 \\ 489270 \end{array} \qquad 412^2= \begin{array}{r} 412 \\ 12 \\ \hline 424\times 4 \\ \hline 1696\mid 44 \\ 1\ \end{array}$$

Note Proceed exactly as before (§§ 51 to 53), but multiply by the *boundary digit before* appending the product of the distances. The place rules in §§ 51 to 53 apply here also.

The proof for the mathematician goes like this: let s be the boundary (900 in the first example), e and d the distances (5 and 7) of the numbers A and B from the boundary, then

$$\begin{aligned} AB&=(s-e)(s-d)=s(s-e-d)+ed \\ &=s(A-d)+ed=s(B-e)+ed. \end{aligned}$$

If s is a unit-boundary, this gives the rules of §§ 51 to 53. If A and B enclose the boundary (last example but one), e and d have opposite signs, so the product ed must be subtracted, as in § 53.

Exercises

(*a*) 42×43, 78×79, 89^2, 62^2. (See also §§ 40, 49, and 61.)

(*b*) 395×391, 782×796, 4992^2.

(*c*) 203×207, 611×607, 925×916.

(*d*) Enclosing numbers (§ 53): 807×798.

Apart from its surprising simplicity, this method offers great advantages for numbers near a boundary. It can be applied more often than one would think. As you see from the examples, you can go quite a long way from the boundary, particularly if you have a few tricks up your sleeve.

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Devices for written multiplication

§ 56

(1) **Factor near a boundary.** If one number differs from a boundary only by a U digit, **multiply by the boundary and subtract the U-multiple:**

$$\begin{array}{r} \underline{786 \times 694}\ (694=700-6) \\ 550200 \\ \underline{-4716} \\ 545484 \end{array} \qquad \begin{array}{r} 786 \times \underline{994}\ (994=1000-6) \\ \underline{-4716} \\ 781284 \end{array}$$

Those who know the medium multiplication table (§ 41(*a*)) can multiply by 687 (=700 − 13) or 5986=(6000 − 14).

Exercises (1) 347859 × 395, 37809 × 997, 7438 × 386.

(2) **Use of partial answers.** One can often use a partial answer as a *preliminary*:

$$\begin{array}{r} \underline{657 \times 84} \\ 2628 \times 2 \\ \underline{5256} \\ 55188 \end{array}$$

2628 × 2 is the same as 657 × 8, since 2628=657 × 4. This method is somewhat risky, as an error in the partial answer is carried through the whole exercise. However, for safety there is always the test.

Don't miss attractive short cuts like these.

$$(a)\ \begin{array}{r} \underline{5673 \times 357} \\ 39711 \quad \times 5 \\ \underline{198555} \\ 2025261 \end{array} \qquad (b)\ \begin{array}{r} \underline{5673 \times 1236} \\ 68076 \quad \times 3 \\ \underline{204228} \\ 7011828 \end{array}$$

(*a*) I see that 35=7 × 5; first multiply by 7, then the partial answer by 5.

(*b*) 36=12 × 3; first multiply by 12 H, then the partial answer by 3, moving two places to the right.

Such possibilities occur more frequently than you would think, if you use your eyes.

Note For safety always write the partial answer in the right place relative to the original number. The number 5673 fixes the places, so that in example (*b*) the 12-product begins as H under the 6 H.

For the teacher. The argument over which end to begin the multiplication is superfluous: both methods must be mastered. What is important is that one knows which unit one is dealing with. This is most sensibly determined by proper alignment under the original number. Compare the following paragraph (3). (In those cases, as in §4, where the working out is omitted, this alignment is not used, for the sake of a neater appearance; likewise, it is not used if there is an advantage in another method as in § 32.)

Exercises (2) 3812 × 642, 629 × 63, 3812 × 248, 6852 × 3015.

(3) **Factor with a 1. If one number contains a 1, its partial product is there already:** it is the other number.

$$\begin{array}{r} \overline{486} \times \underline{71} \\ \underline{3402} \\ 34506 \end{array} \qquad \begin{array}{r} \overline{3875} \times \underline{712} \\ 7750 \\ \underline{27125} \\ 2759000 \end{array} \qquad \begin{array}{r} \overline{4295} \times \underline{134} \\ 12885 \\ \underline{17180} \\ 575530 \end{array}$$

Note (1) If you put a line under the number to be multiplied (486), you may forget to add it in at the end. It is better to put the line *over* the number, then you *see* that all numbers between the lines have to be added.

(2) Write the numbers in *exactly* the right place (2nd example).

(3) Always estimate and test.

A better method for examples 2 and 3: multiply by 12 and 13.

Exercises (3) 483 × 531, 605 × 712, 327 × 124.

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Factor multiplication

§ 57

By this I mean the multiplication of two numbers, not in their existing form, but *split up into factors* and *combined* in another way which is more convenient, particularly for mental work. A simple example will show what I mean.

$$\begin{array}{ll} 42 \times 35 & \text{now } 42 = 2 \times 21 \\ & \phantom{\text{now }} \underline{35 = 5 \times 7} \end{array}$$

$$42 \times 35 = 2 \times 21 \times 5 \times 7 = (21 \times 7) \times 10 = 1470.$$

Splitting off the factors 2 and 5 has given me the *easy* factor 10 and so simplified the work.

RULE **By splitting up one or both numbers into factors, try to find an easier factor.**

Out of the many possibilities, here are **two hints.**

(*a*) **Try to find easy factors such as 10, 11, 15, 25, 125.**

$$\begin{aligned} 86 \times 45 &= (43 \times 9) \times 10 = 3870; \text{ or} \\ &= (86 \times 15) \times 3 = 1290 \times 3 = 3870 \\ 35 \times 65 &= (7 \times 13) \times 25 = 91 \times 25 = 2275 \\ 37 \times 44 &= 37 \times 4 \times 11 = 148 \times 11 = 1628. \end{aligned}$$

(*b*) Try to find a factor near a boundary.

$$17 \times 6 = 102,\quad 35 \times 3 = 105, \text{ and so on.}$$
$$78 \times 17 = 13 \times 6 \times 17 = 13 \times 102 = 1326$$
$$42 \times 35 = 14 \times 3 \times 35 = 14 \times 105 = 1470$$
$$56 \times 27 = 56 \times 3 \times 9 = 1680 - 168 = 1512 \text{ (§ 33).}$$

There is a long list of such numbers which are near to a boundary; note particularly the neighbours of 100.

$8 \times 13 = 104$ $7 \times 15 = 105$ $6 \times 17 = 102$ $3 \times 34 = 102$
$8 \times 12 = 6 \times 16 = 96$ $7 \times 14 = 98$ $3 \times 33 = 99$ $7 \times 11 \times 13 = 1001$
$17 \times 59 = 1003$ $7 \times 29 = 203$ $7 \times 43 = 301$ $3 \times 23 \times 29 = 2001$

Our tricks of §§ 28, 30–32 and 36 are really just applications of this rule. For splitting into factors see § 76.

Exercises can be found throughout the book.

Note If you have to multiply the factor near a boundary, for example 105, by a *two-digit* number (17 or 24), you can do it from the *left* (17′85 and 25′20), since the zero simplifies the work enormously. Remember this for all **three-digit numbers with a zero in the middle**, as for example 603×9 or 603×18. Write them down straight away from the left 54 | 27 and 108 | 54. You can easily carry over if necessary: $708 \times 17 = 120 \mid 36$.

For the teacher. Splitting up and recombining is a useful way of dealing with numbers and getting to know their *peculiarities*. To the expert the series of numbers lined up in the order 1, 2, 3, . . . is not a rank of apparently identical number-soldiers: for him each one has its own individuality, its own 'character'. Soldier 17 is quite different from soldier 16, although they are in direct contact; they employ their talents in quite different ways. The experienced calculator must get to know the *look* of numbers, so that each one can be used correctly. For this purpose there is hardly a more effective method than splitting up and recombining. In this lies the whole secret of the extraordinary accomplishments of the so-called number magicians, men with an extraordinarily well-developed number sense.

These hints and examples are nothing more than a stimulation to solve others for yourself. But no schoolroom should be without such stimulation. And if a youngster works out 33×7 by means of the 11-trick: $21 \times 11 = 231$, he may not have saved much work, but he will have gained in power of observation and familiarity with numbers.

* *

Squares

§ 58 SURVEY OF THE METHOD

If a number is multiplied by itself, 5 × 5, it is written more briefly 5^2 and read: '5 to the power 2', or '5 squared'.

Why 'squared'? A square table-top, 34 in. long, and therefore 34 in. wide, has an area of 34 × 34 sq. in. 34 × 34 or 34^2 is therefore the measure of the area of a square of side 34; hence one reads 34^2 as '34 squared'.

And how is 34^2 calculated?

We have the following **general methods.**

(1) the *double-digit trick*, which always applies for squares, § 46

(2) the method of *numbers near a boundary*, § 55

(3) the method of *numbers with like tens*, § 40.

Exercises Work out the squares of § 59 by these methods.

However for squares there is a set of attractive and surprising 'square'-methods, which we shall learn now, the first two for written work (§§ 59 and 60), then for squares from 1 to 100 mentally, ending with some particular squares (§ 62–64).

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§ 59 WRITTEN METHOD FOR TWO-DIGIT NUMBERS

Suppose 46^2 is required.

$$\begin{array}{r} 4 \mid 6 \\ \hline 16\;\;36 \\ 4\;8\;\; \\ \hline 21\;\;16 \end{array}$$

(1) **Split the number into T and U 4 | 6 and form the square of each: 16 | 36.**

(2) **double one digit (T or U), and multiply by the other: 8 × 6 or 12 × 4 = 48 and write the product as a T** (and therefore one place *to the left*);

(3) **add these together: 2116.**

Squares must be written as *two*-digit numbers (04, not 4)

$$42^2 \qquad 8\times 2 \text{ or } 4\times 4 \qquad \begin{array}{r} 4|2^2 \\ \hline 16\;04 \\ 1\;6\;\; \\ \hline 17\;64 \end{array}$$

In this way quite difficult exercises can be solved quite easily. Admittedly we are really using the well known formula $(a+b)^2 = a^2 + 2ab + b^2$, but the way of writing it down makes the formula easy to apply.

Exercises (*a*) 27^2, 33^2, 64^2, 76^2, 92^2 (*b*) 115^2, 117^2, 253^2, 402^2.

For the teacher. For a number with a high U many people recommend one to relate the number to the next *higher* boundary number: $68^2 = (70-2)^2$, and then use the formula $(a-b)^2 = a^2 - 2ab + b^2 = 4900 - 280 + 4$.

I do not recommend this way, because (1) it works with numbers *other* than those given; if 68^2 is given, it is a mental burden to work with the equivalent $(70-2)^2$; and (2) it involves *subtraction*, and again the numbers are not the given ones. Addition is always easier and safer than subtraction. We see here once again that a formula is not a trick. What distinguishes a trick is the *mental* simplification.

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§60 THE 'FUNNEL' FOR NUMBERS WITH THREE OR MORE DIGITS

The squares of larger numbers can be written down in the following attractive way:

(a)

$$\begin{array}{c} 642^2 \\ abc \\ \hline 361604 \\ 4816 \\ 24 \\ \hline 412164 \end{array}$$

(b)

$$\begin{array}{c} 372589^2 \\ abcdef \\ \hline 094904256481 \\ 4228208\mathbf{1}44 \\ \mathbf{1}2703290 \\ 3\mathbf{1}1236 \\ 4926 \\ 54 \\ \hline 138822562921 \end{array}$$

(c)

$$\begin{array}{c} 4562^2 \\ \hline 16253604 \\ 406024 \\ 4820 \\ 16 \\ \hline 20811844 \end{array}$$

(1) Write in the **first row** the *square* of each digit as a *two*-digit number (04).

(2) Now form the *double-product* of the **first** digit a with every other, thus $a \times b \times 2$; $a \times c \times 2$; $a \times d \times 2$... (in example (*a*) $6 \times 4 \times 2 = 48$, $6 \times 2 \times 2 = 24$) and write them as *two*-digit numbers, each *below* and *one* place to the *right* of the previous one. You will obtain the **first slope,** in example (*a*) (36)48 24, in example (*b*) (09)42 12 30 48 54.

(3) Form the *double product* of the **second** digit b with each subsequent one, thus $b \times c \times 2$, $b \times d \times 2$, and so on, and write them as two-digit numbers on the steps of the first slope. You will obtain the **second slope** (in example (*a*) 16 from $4 \times 2 \times 2$, in example (*b*) 28 70 112 126 from $7 \times 2 \times 2 = 28$, $7 \times 5 \times 2 = 70$, and so on). The H must be added to the previous digit; the numbers in heavy type in the funnel show when this has happened. Example (*a*) is now almost finished: add the rows together and you have the answer. In example (*b*) you must form all the other double products.

You can, if you prefer, complete each row in turn. For example in (*c*):
Row 1: squares; *row* 2: 1-steps 4×5; 5×6; 6×2; *row* 3: 2-steps 4×6; 5×2; *row* 4: 3-step 4×2.

Each product is naturally doubled. 1-step means a number multiplied by its neighbour, 2-step means jump over the neighbour, 3-step means jump over two neighbours, and so on. Work the examples by both methods, and adopt the one which appeals to you most.

The proof follows from the formula

$$\begin{aligned}(a+b+c+\dots)^2 = {} & a^2+b^2+c^2+\dots\\ & +2ab+2bc+\dots\\ & +2ac+2bd+\dots\\ & +2ad\dots\end{aligned}$$

Exercises 322^2; 4864^2; 85348^2

Note The test should be used here also. For (*c*) it is: T ($8\times8=64$; 1 | 1). Compare § 12.

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§ 61 THE SQUARES OF NUMBERS FROM 1 TO 100 MENTALLY

Every calculator knows how often the squares of two-digit numbers are necessary in advanced work. Anyone who, in addition to halving, doubling, adding, and subtracting two-digit numbers, can also square them, has a claim to the title of master-calculator.

We split the squares into three groups:

A	*B*	*C*
1–25	25–50–75	75–100–(125)
Beginning-	middle-	end-squares

we shall call them.

(*A*) Beginning-squares of numbers from 1 to 25

should be known *by heart*. Naturally this holds particularly for those from 11 to 25. With a little practice you can soon remember them as easily as those from 1 to 10.

$11^2=121$
$12^2=144$
$13^2=169$
$14^2=196$
$15^2=225$
$16^2=256$
$17^2=289$
$18^2=324$
$19^2=361$
$20^2=400$
$21^2=441$
$22^2=484$
$23^2=529$
$24^2=576$
$25^2=625$

You can quickly work out the squares from 11 to 19, when you meet them for the first time, by § 39; then 22^2 by § 34(*c*) and 24^2 by § 30. For 5-squares see § 62 (*A*).

The teacher can use these squares in the fourth or fifth year of arithmetic. They can first be worked out by any method, and then learnt by heart like words of a foreign language. This is very little compared with the material which has to be learnt in other subjects!

Teachers probably fight shy of introducing the idea of *powers* or *squares* in the lower classes. However, it is not a question of working with powers here; the term 'square' is an *abbreviation* for a product of two equal factors, so the objection is untenable. Unit-boundaries can very well be treated as powers of 10 in the fifth year, thus making work with areas much clearer: 1 sq. km. is $10^2 \times 10^2 \times 10^2 = 10^6$ sq. m. It can be worked out as $100 \times 100 \times 100$, but then the answer comes out as 1 million! Compare § 90.

(*B*) The middle squares of numbers from 25 to 75

group themselves about the number 50. The squares of numbers in the fifties (54^2, 57^2, . . .) can then be worked out surprisingly simply.

RULE **Add the U to 25, and append the U-square as a two-digit number.**

Examples 54^2; $25+4=29$, then $4^2=16$, giving 29′16.
Similarly $52^2=27'04=2704$. 57^2; $25+7=32$; 32′49.

Now you can go further from the mean 50, and use the beginning-squares,

(1) above: 62^2; $25+12=37$, so 38′44. $68^2=46'24$.

(2) below: 48^2; $25-2=23$, so 23′04; also $46^2=21'16$. 37^2; $25-13=12$, then $13^2=169$ as a two-digit number gives 13′69.

RULE **Add the 50-distance of the number to 25 (subtract from 25 in the second case) and append its square as a two-digit number.**

Proof by means of the formula in § 59:

$$54^2=(50+4)^2=2500+400+16=(25+4)\times 100+16;$$
$$48^2=(50-2)^2=2500-200+4=(25-2)\times 100+4=2304.$$

Exercises, of which you can test the answers by writing them out as in § 59:
(1) 56^2, 59^2, 61^2, 64^2, 69^2, 72^2. (2) 47^2, 42^2, 39^2, 33^2, 28^2, 26^2.

Hint. In the second case you can find the first two digits of the square in another way, as you can see in the examples 48^2 and 37^2:

$$25-2=48-25=23 \quad \text{and} \quad 25-13=37-25=12.$$

This then gives a further

RULE **Subtract 25 from the given number and append the square of its 50-distance as a two-digit number.**

This rule holds also for the squares of numbers over 50:

$$62^2;\quad 62-25=37,\ \text{then } 12^2,\ \text{so } 3844.$$

Both rules therefore apply to all middle-squares. I prefer the first.

(*C*) End-squares of numbers from 75 to 100

and then on to 125 group themselves about the boundary 100 and can be worked out mentally as in §§ 51 and 52:

$$97^2;\quad 97-3=94,\ \text{then } 3^2=9,\ \text{so } 94'09.$$

RULE **Subtract from the number its 100-distance and append the square of this distance as a two-digit number.**

$$92^2;\quad 92-8=84,\ \text{so } 84'64;\quad \text{also } 88^2=77'44.$$

Correspondingly as in § 52: 106^2; $106+6=112$, then 36, so 11236.
123^2; 146, then $23^2=529$, so 15129.

Exercises (3) 93^2, 98^2, 87^2, 82^2, 77^2. (4) 103^2, 107^2, 115^2.

Summary. If you know the beginning-squares up to 25, you can arrange the middle-squares from 25 to 75 about the number 50 and the end-squares from 75 to 125 about 100, and so *work out the squares of all numbers from 1 to 125 in your head.*

For the experienced calculator. Squares of three-digit numbers are found by using § 59:

$$452^2\quad \begin{array}{rr} 4\ | & 52^2 \\ \hline 16 & 2704 \\ 4 & 16 \\ \hline 20 & 4304 \end{array} \quad \text{or} \quad \begin{array}{rr} 45\ | & 2^2 \\ \hline 2025 & 04 \\ 18 & 0 \\ \hline 2043 & 04 \end{array} \qquad 593^2\quad \begin{array}{rr} 5\ | & 93 \\ \hline 25 & 8649 \\ 9 & 30 \\ \hline 35 & 1649 \end{array} \quad \text{or} \quad \begin{array}{rr} 59\ | & 3 \\ \hline 3481 & 09 \\ 35 & 4 \\ \hline 3516 & 49 \end{array}$$

(*D*) A general rule for all squares

$$48^2=(50\times 46)+2^2=2300+4=2304 \qquad 63^2=(60\times 66)+3^2=3969$$
$$46-48-50 \qquad\qquad 60-63-66$$

RULE Transform the square (48^2) into a *product of two mirror-numbers* (50×46), of which one is a *complete* T (50), and add the square (2^2) of the distance (2) from the mirror. The mirror is the given number (48). (For mirror-numbers see § 54.)

This rule is recommended for those who do not know the higher beginning-squares by heart; they can then work more easily

$$76^2=(80\times 72)+4^2=5760+16=5776.$$

The proof (for experts) follows from the formula $(a+b)(a-b)=a^2-b^2$, which, when solved for a^2, gives our rule: $a^2=(a+b)(a-b)+b^2$. Here a is the mirror, b the distance from it.

Exercises Those in (*B*) and (*C*) above.

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§ 62 PARTICULAR SQUARES

There is a whole series of numbers which can be squared quickly in a surprising way. First we have

(*A*) the 5-squares,

which are the squares of numbers ending in 5, such as 15^2, 35^2, 125^2:

$15^2=225=1\times 2 \mid 25$ $\quad$ $35^2=1225=3\times 4 \mid 25$
$25^2=625=2\times 3 \mid 25$ $\quad$ $75^2=5625=7\times 8 \mid 25$

RULE **For a 5-square multiply the T by the T increased by 1 and append 25.**

This is a wonderful trick, which can be used for an estimate: $73\times 75 \doteq 75\times 75=75^2=5625$. In no time we have a rough value.

It is bewildering if done mentally: $115^2=132 \mid 25$, or by § 39: $165^2=27225$. It is simple to work out high squares: $485^2=48\times 49 \mid 25$, which can be done easily with the DD-trick of § 46: 235225. The answer can be written down straight away!

Now work out, to the amazement of your audience, these

Exercises (1) 45^2, 95^2, 125^2, 245^2. (2) 255^2, 495^2, 325^2, 585^2.
(3) 635^2, 135^2, 185^2, 145×146.

The proof. The 5-squares are a special case of § 49 (1). It can also be seen like this:

$$\begin{aligned}75^2&=[(7\times 10)+5]^2=[(7\times 10)+5][(7\times 10)+5]\\&=(7\times 7\times 100)+(2\times 5\times 7\times 10)+25\\&=(7\times 7\times 100)+(7\times 100)+25=(7\times(7+1)\times 100)+25\\&=(7\times 8\times 100)+25.\end{aligned}$$

The general proof for *any* number ending in 5 goes like this, or very simply as a special case of the formula of § 39.

(*B*) 1-squares

are squares of numbers which end in 1, such as 31^2, 71^2, 121^2, 731^2.

Now $(T+1)^2=T^2 \qquad +2T \qquad +1$
$31^2=(9\times 100)+(2\times 3\times 10)+1=961$, which gives the

RULE **Between the square of the T ($3^2=9$) and 1 write twice the T (6) as a one-digit number.**

Examples (1) $41^2=16\ 8\ 1$ with carry-over
$$\begin{array}{r} 81^2=64\ 1 \\ \underline{16\ \ } \\ 6561 \end{array}$$

(2)
$$\begin{array}{r} 171^2=289\ 1 \\ \underline{34\ \ } \\ 29241 \end{array}$$

(3) 681^2
$$\begin{array}{r} \underline{68\ \ 1^2} \\ 4624\ 1 \\ \underline{136\ \ } \\ 463761 \end{array}$$

(4)
$$\begin{array}{r} \underline{43\ 1^2} \\ 1849\ 1 \\ \underline{86\ \ } \\ 185761 \end{array}$$

Note (1) For two-digit numbers the carry-over (if any) is always 1; you notice this for numbers from 51 upwards and add it to the square straight away.

(2) For numbers from 111 to 191 you have the large multiplication table (§ 39), if you do not know the squares of numbers from 11 to 19 by heart; see § 61 (*A*).

(3) For higher numbers, such as 681^2, use the squares of § 61.

Exercises (4) 61^2, 91^2, 11^2, 121^2, 141^2, 251^2, 741^2 (compare § 46 (1)).

(*C*) Squares of numbers near a boundary

are squares of numbers near a unit-boundary, such as 994^2 and 1012^2, and with the help of §§ 51 and 52 can easily be found by the

RULE **Add the distance of the number from the boundary to the number itself, and append the square of this distance, as a number with as many digits as the boundary has zeros.**

If the number lies *below* the boundary (994), the distance is negative (-6), that is, it must be subtracted ($994-6=988$). Compare the end-squares of § 61 (*C*).

Examples 994^2; distance 6, since $994+6=1000$; giving 988036.
$1012^2=1024144$ $9986^2=99720196$ $1025^2=1050625$.

Exercises (5) 996^2, 9991^2, 1011^2, 1023^2, 412^2 and 785^2 by § 55.

For experienced calculators I introduce here the little-known trick of

(*D*) 25-squares,

which are squares of numbers ending in 25, such as 325^2 or 625^2.

RULE **Add to the square of the H half the H, multiply the sum by 10, and append 625.**

Examples 625^2; H-square $6^2=36$; H-half 3 ($=6\div 2$) added to 36 is 39; times 10 gives 390; then 625, so 390625. In the same way

325^2; $9+1{\cdot}5=10{\cdot}5$; 105625. 1525^2; $225+7{\cdot}5=232{\cdot}5$; 2325625.
1225^2; $144+6=150$; 1500625. 5825^2; $3364+29=3393$; 33930625.
(§ 61 (*B*))

Proof.
$$\begin{aligned} 625^2=(600+25)^2 &= 600^2+(50\times 600)+25^2 \\ &= [(600+50)\times 600]+625 \\ &= (65\times 6\times 1000)+625=65\times 6 \mid 625. \end{aligned}$$

Like this it is not difficult to work out; however the connection is clearer if I recast it in the form:

$$\begin{aligned} 65\times 6 \mid 625=6{\cdot}5\times 6\times 10 \mid 625 &= (6+\tfrac{1}{2})\times 6\times 10 \mid 625 \\ &= [6^2+(\tfrac{1}{2}\times 6)]\times 10 \mid 625 \end{aligned}$$

and this is the above rule!

Exercises (6) 425^2, 825^2, 1425^2, 2425^2, 725^2, 1325^2. With a little practice they can be done mentally, particularly if you give the solution in the form in which the calculation is carried out, for example 625^2; '36, 39-0-625'.

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§ 63 THREE MORE PARTICULAR TYPES OF SQUARE

are introduced here, to show how for some numbers quite a simple trick can be devised:

(*A*) The squares of numbers from 400 to 600

lie about the mean 500, and if we think of the middle-squares of § 61(*B*) we can imagine a similar relation. In this case we have the corresponding, surprisingly simple

RULE **Subtract 250 from the number and append the square of the 500-distance (as a three-digit number).**

$$513^2 \quad \begin{array}{ll} 513 & 13 \\ -25 & \\ \hline \multicolumn{2}{l}{263 \mid 169} \end{array} \qquad\qquad 487^2 \quad \begin{array}{ll} 487 & 13 \\ 25 & \\ \hline \multicolumn{2}{l}{237 \mid 169} \end{array}$$

537^2; $287 \mid 1369$; 288369 (§ 61 (*B*)). 412^2; $162 \mid 7744$; 169744 (§ 61 (*C*)).

Instead of subtracting 250 from the number (513), one can of course subtract 25 from its tens (51).

The proof corresponds exactly to that of § 61 (*B*), since the limits can be pushed to 250 below and 750 above. Similarly the rule can be extended to the squares of numbers round about 5000, 50000, and so on.

Exercises (*a*) 516^2, 552^2, 489^2, 425^2, 5017^2.

(*B*) Squares of numbers consisting of threes or nines

$33^2=1089$ $\qquad$ $99^2=9801$
$333^2=110889$ $\qquad$ $999^2=998001$
$3333^2=11108889$ $\qquad$ $9999^2=99980001$ and so on.

You can *see* the rule for yourself. You can discover the proof if you work out $333^2=333\times 333$ by ordinary methods.

(*C*) Squares of 0-mirror numbers

303^2	404^2	707^2	$12\ 0\ 12^2$	$25\ 0\ 25^2$
9 18 09	16 32 16	49 98 49	144 288 144	625 250 625
				1

RULE **Write twice the square of the digit between the square written twice.**

The squares and their doubles are written with *one more* place than the digit-group itself has; so in the first three examples they are written with two places, in the last two examples with three (in the last example 626250625).

The proof is obtained easily as in § 59.

Exercises (*b*) 505^2, 202^2, 909^2, 13013^2, 18018^2, 57057^2 (§ 61 (*B*)).
What rule can you find for 4004^2?

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§ 64 NEIGHBOURING SQUARES

I call those which lie near an easy (5-, 1-, 10-, 25-) square or one which happens to be known; for example 37^2 is a neighbouring square of 35^2 or 40^2. Can 37^2 be worked out by using one or other of these? Yes, very easily by means of the

Neighbouring formula: $(a+n)^2=a^2+[a+(a+n)]\times n$.

Here a is the 'easy' number 35, n is its distance 2 from the given number: $a+n=37$. Then the formula gives

$$\begin{aligned}37^2&=35^2+(35+37)\times 2\\&=1225+144=1369.\end{aligned}$$

If the 'easy' square 40^2 lies *above* the given one 37^2, we call the distance negative, so we must *subtract* from 40^2 to get 37^2:

$$\begin{aligned} 37^2 &= 40^2 - (40+37) \times 3 \\ &= 1600 - (77 \times 3) = 1369, \text{ as before.} \end{aligned}$$

This formula is very useful for finding the **immediate neighbour**, in which case $n = 1$:

$$(a+1)^2 = a^2 + [a + (a+1)], \text{ which gives the}$$

RULE **Add to the known square (or subtract from it) the sum of the two numbers.**

$$36^2 = 35^2 + (35+36) = 1225 + 71 = 1296. \text{ Correspondingly}$$
$$34^2 = 35^2 - (35+34) = 1225 - 69 = 1156.$$

The proof of the neighbouring formula follows from the well-known formula

$$(a+n)^2 = a^2 + 2an + n^2$$

neatly rewritten in the above form.

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§65 MULTIPLICATION IN RETROSPECT

Squares have disclosed many attractive and surprising relations between numbers. Although there are many more or less intricate rules which could still be given, those which we have met with up to now give a good foundation. Squares bring us to the end of the work on multiplication. If we look back, we find a rich harvest. There will be few examples for which none of our 'tricks' will work, whether they be in the categories of particular numbers, those of cross- or column-multiplication, numbers near a boundary, or squares.

One thing to remember when studying: go forward *slowly*, and only start a new trick when you have used the old ones in your everyday reckoning for a while so that they become automatic. A trick which can be remembered only with difficulty is not a trick at all. In this way alone will you escape the danger of being discouraged by the variety of different ways of tackling a sum. If you have really mastered a method, the variety will not frighten you: on the contrary, it will spur you on to new voyages of discovery.

Don't forget the *test* at the end of your work (§§ 2 ff.). Also, now that you have mastered a number of tricks, you should become familiar with *estimates* (§ 89). *Before* doing a sum, always mark out the area within which the answer is hiding!

DIVISION

Division by particular numbers

It is often an advantage to know how to divide by certain particular numbers; here again we meet the numbers 5, 25, 125.

Division is certainly the most unpleasant process. In former times people positively struggled with it. For that reason, anything which simplifies the process is very important; so are the *test* (§ 9) and *estimate* (§§ 89 ff.).

* *

§ 66 DIVISION BY A UNIT-BOUNDARY

It is very easy to divide by 10, 100, 1000, and so on, because our number-system is based on the number 10.

RULE **Cut off as many places as the divisor has zeros.**

$$483 \div 10 = 48{\cdot}3 \qquad 483 \div 1000 = 0{\cdot}483$$
$$\div\, 100 = 4{\cdot}83 \qquad \div\, 100000 = 0{\cdot}00483$$

Hint. Compare § 90 (For the teacher).

* *

§ 67 DIVISION BY 5

$20 \div 5 = 4$
$42 \div 5 = 8$ remainder 2
$63 \div 5 = 12$ remainder 3

RULE **To divide a number by 5, double the T and divide the rest by 5.**

Proof. 5 is contained twice in 10.

If the U is greater than or equal to 5, it must be divided by 5.

$78 \div 5$; $7 \times 2 = 14$; $8 \div 5 = 1$ remainder 3, so $14 + 1 = 15$, **15 remainder 3.**

$689 \div 5$; $68 \times 2 = 136$; $9 \div 5 = 1$ remainder 4, and so 137 remainder 4.

For simplicity split off the T by a stroke above:

$$68'9 \div 5 = (136)\ 137\|4 \qquad 14832'6 \div 5 = 29665\|1.$$

As a decimal the answer can be obtained more simply:

cut off one place from the number and double it!

The cutting-off of one place is equivalent to division by 10 (§ 66):

689 ÷ 5 = 68·9 × 2 = 137·8;   384 ÷ 5 = 38·4 × 2 = 76·8.

I prefer to cut off the place first and then double, because the partitioned number 38·4 is easier to deal with than 384 (compare § 29 (*A*)).

Exercises See § 69.

☆ ☆

§ 68 DIVISION BY 25

Since there are four 25-packets in every 100, and so 2 × 4 in 2 H, and so on, we have the

RULE **To divide a number by 25, multiply its H by 4 and divide the rest by 25.**

Examples 4′00 ÷ 25 = 4 × 4 = 16;
4′25 = (4 × 4) + 1 (from 25 ÷ 25) = 17;
4′58 = (4 × 4) + 2 remainder 8 = 18 remainder 8.

Only the T and U, that is, the 2-place ending of a number, have to be *divided* by 25, so the quotient can only be 1, 2 or 3: the remainder can be any number from 1 to 24. The important thing is that *division* is replaced by *multiplication* of the H. The numbers 417, 67873, 577 are divided by 25 like this:

4′17 ÷ 25	678′73 ÷ 25	5′77 ÷ 25	6′85 ÷ 25
16 | 17	2712 | + 2 rem. 23	23 | 2	27·4
16·68	2714 | 23	23·08	
	2714·92		

The vertical stroke splits off the remainder. The H is split off to start with by means of a stroke above (as in 4′17).

As a decimal: multiply the remainder by 4 and write it as a two-digit number after the decimal point.

The method is often taught in the form: first cut off two places and then multiply by 4; 5·77 × 4. I prefer the given way, which is safer and quicker.

Exercises See § 69.

☆ ☆

§ 69 DIVISION BY 125

By this time we should know the

RULE **Multiply the Th by 8 and divide the rest by 125,**
since 125 is contained 8 times in 1000, 2 × 8 times in 2000, and so on. Thus

$$\begin{array}{l} 121'098 \div 125 \\ \hline 968 \mid 98 \\ \hline 968{\cdot}784 \end{array} \qquad \begin{array}{l} 302'257 \div 125 \\ \hline 2416 \mid +2 \text{ remainder } 7 \\ \hline 2418 \mid 7 \end{array} \quad = 2418{\cdot}056.$$

Only the 3-digit ending of a number has to be divided; for this, it is a good thing to know the 125-multiples: 125, 250, 375, 500, 625, 750, 875. To start with, split off the Th by means of a stroke above (as in 121'098).

As a decimal: multiply the remainder by 8 and write it as a 3-digit number after the decimal point.

Exercises Divide with remainder and as a decimal

(1) by 5: 82, 174, 327, 637025.
(2) by 25: 317, 325, 358, 416, 1278, 74021.
(3) by 125: 4250, 32127, 458278, 9739874.
(4) by 5, 25, and 125: 982403754.

Do the smaller exercises partly, and then completely, in your head.

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§ 70 SHORT DIVISION,

as it is called when the remainders are not written down, causes youngsters considerable difficulty, even with 1-digit divisors. The difficulty disappears surprisingly:

(1) if the partial answers are written *under* the number to be divided (9 1 and so on in the example below), as we did in §§ 68 and 69, and as I recommend emphatically;

(2) if each remainder is *written* small and low down *in front of the following digit*. The number which is now to be divided can then be seen. This is safer, because the work can confidently be broken off to take a breath. Calculating and retaining numbers are two different things, and all too often a boy becomes disheartened not because of the calculation, which he enjoys on the whole, but because of the pressure on the memory. Admittedly he should in time increase his powers of memory.

So the solution of the division 643875 ÷ 7 goes like this:

$$\begin{array}{l} 6\;4\,{}_1 3\,{}_6 8\;7\;5 \div 7 \\ \;\;\;9\;\;1\;.\;\;.\;\;. \end{array} \qquad \begin{array}{l} 6\;4\,{}_1 3\,{}_6 8\,{}_5 7\,{}_1 5 \div 7 \\ \;\;\;9\;\;1\;\;9\;\;8\;\;2 \mid 1 \end{array}$$

that is 64 ÷ 7, 13 ÷ 7, 68 ÷ 7, and so on.

In this way division by 11, 12, and so on can easily be done by short division.

$$\begin{array}{l} 6\ 4{}_{4}3{}_{7}8{}_{6}7{}_{7}5 \div 12 \\ \quad 5\ 3\ 6\ 5\ 6 \mid 3 \end{array}$$

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§ 71 FURTHER METHODS FOR DIVISION

Division by particular numbers $5=\frac{10}{2}$, $25=\frac{100}{4}$, $125=\frac{1000}{8}$ is up to now our only 'trick' for division. For example, dividing by 15 by expressing it as $\frac{3\times 10}{2}$, thus $428\div 15=428\div\frac{3\times 10}{2}=\frac{428\times 2}{3\times 10}=86{\cdot}6\div 3=28{\cdot}53333\ldots$ is not a trick.

Also the usual school method of working out $63857\div 48$ is not very simple, and just as it once presented insuperable problems to the calculators of the Middle Ages so even for our more experienced calculators it has not lost its torments: the everlasting testing, multiplying and subtracting.

We shall now learn two methods, *factor-division* and *boundary-division*, with which most divisions can be carried out surprisingly simply.

The two ways of simplifying the division $63857\div 48$ are:

(1) The divisor 48 is split into *factors*, $48=6\times 8$ or $3\times 4\times 4$, and one divides by the factors: this is **factor-division**.

(2) Instead of dividing by 48 one divides by its neighbouring *boundary*, here 50: this is **boundary-division**.

In both methods division by a number with *more than one* digit is replaced by division by a smaller number, usually a *one*-digit number. This is the basic idea.

Together they give a significant simplification of the existing division. Factor-division at least should be introduced in schools, and should replace the old long-winded methods where possible.

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Factor-division

§ 72 THE GENERAL IDEA

Since $36=4\times 9$, the sum 8257×36 can be worked out like this:

$$\begin{array}{r} 8257\times 4 \\ \hline 33028\times 9 \\ \hline 297252 \end{array}$$

Is this really an advantage? No, not really, since I can multiply by 3 and 6 more easily than by 4 and 9. Admittedly I don't have to add.

But *conversely*: if I have to divide 297252 by 36, is it an advantage to divide in turn by 4 and 9 instead? Yes, it certainly is, because 4 and 9 are *one*-digit numbers, and I can divide by them quickly and easily. This is the main point of the method.

Instead of dividing by a two-digit number, for example, divide by its one-digit factors.

This is a quite unexpected *simplification*. If you doubt it, just divide by 36, with all the division, multiplication, subtraction—and then, on the other hand, two short divisions which you can do in your head. This transformation plays an important part in percentages (§ 97(*c*)).

You may object that this is all very well if you know that 36 goes into 297252 without remainder. But what if there is a *remainder*? It's all right, factor-division still works!

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§ 73 FACTOR-DIVISION WITH REMAINDER

The solution of the sum 87 ÷ 35

(*a*) in decimals

87 ÷ 35 (or 5 × 7)
÷ 5 17·4
÷ 7 2·485. . .
Answer: 2·4857142

(*b*) with remainder

87 ÷ 35
÷ 5 17 | 2
÷ 7 2 | (3 × 5) + 2 = 17
Answer: 2 | 17

Method (*a*) Simply divide past the decimal point.

Method (*b*)

(1) Divide 87 by the factor 5, partial answer 17 | 2 (read: '17 remainder 2');

(2) divide 17 by 7, partial answer 2 | 3;

(3) multiply this remainder 3 by the *previous* divisor 5, thus 5 × 3,

(4) and add on the *previous* remainder 2 to give the *total remainder*, thus (5 × 3) + 2 = 17.

(5) Answer: 2 | 17.

Proof for the formation of the remainder:

$$\begin{aligned} 87 &= (5 \times 17) + 2 \\ 17 &= (7 \times 2) + 3 \\ \text{so} \quad 87 &= 5[(7 \times 2) + 3] + 2 \\ 87 &= (35 \times 2) + [(5 \times 3) + 2] \end{aligned}$$

Hint. Begin with the 'easier' divisor. This is usually the smaller.

Example 19605 ÷ 36 (or 4 × 9) = 544 | 21
÷4 4901 | 1
÷9 544 | (5 × 4) + 1 = 21

19605 ÷ 36
÷4 4901·25
÷9 544·583

Always write down the divisor, because you will need it to find the remainder; the remainder is split off by means of a vertical stroke, then you will have a neat, compact form, which says all that is necessary as briefly as possible.

Important application: transformation of fractions into decimals.

$$\frac{8}{35}=\frac{8}{5\times 7}\quad\begin{matrix}1{\cdot}6\\ 0{\cdot}2285\ldots\end{matrix}\qquad\qquad \frac{41}{18}=\frac{41}{2\times 9}\quad\begin{matrix}20{\cdot}5\\ 2{\cdot}27\end{matrix}$$

Exercises (*a*) 176245 ÷ 28, 72, 45, 33.
(*b*) 478295073 ÷ 15, 48, 54.
(*c*) $\frac{7}{15}$, $\frac{19}{42}$, $\frac{53}{33}$.

Note. Division by a **boundary**, 3894 ÷ 70 or 53078 ÷ 400, as it is generally taught, is just factor-division, first by 10 (100) (by § 66) and then by 7 (4):

387 | 4 ÷ 7 | 0 = 55 rem. 24; 530 | 78 ÷ 4 | 00 = 132 rem. 278.

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§ 74 IF THE DIVISOR SPLITS UP INTO SEVERAL FACTORS

our method can of course be applied.

A simple example, which we should normally do by splitting the divisor into 3 × 10 (§ 73, note), will demonstrate the method.

(*a*) as a decimal
46253 ÷ 30 (or 2 × 3 × 5)
÷2 .23126·5
÷5 4625·3
÷3 1541·76...

(*b*) with remainder
46253 ÷ 30
÷2 23126 | 1
÷3 7708 | (2 × 2) + 1 = 5
÷5 1541 | (3 × 3 × 2) + 5 = 23

Note The remainder of each division must be multiplied by the *product* of all divisors already used and added to the *previous* remainder; or briefly the

REMAINDER RULE **Multiply each partial remainder by the product of all previous divisors and add the previous remainder.**

From this follows **method (*b*)**

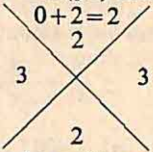
(1) In the division by 2 we get partial remainder 1;
(2) 23126 ÷ 3 leaves partial remainder 2. Previously we had divided by 2, so total remainder = (partial remainder × 2) + 1, so (2 × 2) + 1 = 5;
(3) 7708 ÷ 5 leaves partial remainder 3; product of previous divisors 2 × 3 = 6, previous remainder 5, so total remainder (3 × 6) + 5 = 23. This is the final remainder of the sum.

The proof for the remainder can easily be constructed as in § 73.

One more example:

5480732 ÷ 336 (or 6 × 7 × 8)
÷6 913455 | 2
÷8 114181 | 44
÷7 16311 | 236

The test (§ 9) holds:



The whole division is done in three lines, which can be written down immediately. Admittedly the divisor has to be factorised into 6 × 7 × 8. The divisibility rules (§§ 77 ff.) help in this.

Exercises (*a*) Divide 8597 by 16 in different ways, using 16 = 4 × 4, 2 × 8, 2 × 2 × 4, and 2 × 2 × 2 × 2.
(*b*) 70428 ÷ 63, 44, and 144. Don't forget the test and estimate.
Further exercises in § 77.

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§ 75 FACTOR-DIVISION WITH THE 25- AND 125-TRICKS

(§§ 68 and 69) is a special delight which shows clearly the value of this kind of division:

48732 ÷ 75 (or 3 × 25)
÷25 1949 | 7
÷3 649 | 2 | 57

7053829 ÷ 225 (or 9 × 25)
÷25 282153 | 4
÷9 31350 | 3 | 79

For the reader's benefit I have written in 649 | 2 | 57 the partial remainder of the last division by 3; the answer is, of course, 649 remainder 57. The other answers should be read similarly.

83750253 ÷ 375 (or 3 × 125)
÷125 670002 | 3
÷3 223334 | 3

6328 ÷ 875 (or 7 × 125)
÷125 50 | 78
÷7 7 | 1 | 203

Exercises (*a*) 764524 ÷ 75, and 175 (*b*) 4872034 ÷ 625, and 875.
Work these exercises also by further factorisation.

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Rules for divisibility

§ 76 SPLITTING A NUMBER INTO FACTORS

As the examples of §§ 74 and 75 show, nobody should overlook the superiority of factor-division, particularly in schools.

The method has just one snag: *splitting up the divisor into factors*. How do I know that 336 is $6\times7\times8$? Small numbers like 16, 24, 84 can be factorised even by an inexperienced calculator. Everyone has learnt in school to split up a number into a *product of prime numbers*, that is, numbers which have no further factors (apart from 1 and the number itself).

$$64=2\times2\times2\times2\times2\times2;\quad 78=2\times3\times13;\quad 693=3\times3\times7\times11.$$

To know whether a number has the factor (or 'divisor') 2 or 7, for example, we use the so-called *divisibility rules*. These are rules which tell us, without actually dividing, whether, for example, 693 is 'divisible' by 3, that is, whether 3 goes into 693 without remainder.

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§ 77 THE SIMPLEST RULES FOR DIVISIBILITY

A number is divisible

(1) **By 2—4—8** if its last digit—last two digits—last three digits form a number divisible by 2-4-8: thus **54**—345**616**—76549**728**.

The proof for 4 and 8 follows from rules of §§ 30 and 32.

(2) **By 3 or 9** if the cross-sum is divisible by them. 456; C(456)=15. 861345; C(861345)=27.

Proof in § 8.

(3) **By 5** if the last digit is 0 or 5.

(4) **By 25** if its last two digits are 25, 50, 75, or 00.

(5) **By 15,** for example, if it is divisible by 3 and by 5; **by 24,** if it is divisible by 3 and by 8.

(6) **By 11,** if its ER is zero (§ 13).

Examples $128=4\times32=4\times4\times8$ or $2\times8\times8$, whichever is more suitable. Usually the factorisation need not be continued right down to the prime factors.

$57=3\times19$; $453=3\times$?. The other factor is obtained by dividing by 3, $453\div3=151$, so $453=3\times151$. But 151 cannot be factorised further.

So we have come to a case where our beautiful method of factor-division can break down: namely, when the divisor is a prime number like 43; or when one of the prime factors is large, as in 453. Even so, sometimes the division can be carried out more easily by factors than by the original divisor; it is probably easier to divide by 151 than by 453.

The significance of our method lies in dividing by an 'easier' divisor. A prime number cannot be split up any more; for 'difficult' prime numbers like 47 and 79 factor-division breaks down. We shall learn how we can often divide easily by these numbers in boundary-division in §§ 81 and 87. Now the reader should solve these

Exercises (*a*) $400107 \div 735$ (*c*) $642816 \div 891$
(*b*) $339852 \div 735$ (*d*) $375291 \div 891$.

One can usually manage with these simple rules for divisibility, which are usually taught in school. However, since *factorisation* is particularly attractive to anyone who is keen on numbers, I give in the following §§ 78 and 79 a series of divisibility rules, which will certainly fascinate him because of their peculiarity. Firstly, some for everybody.

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§ 78 DIVISIBILITY OF A NUMBER BY 7, 11, AND 13

We test by means of the 'fabulous'

1001-*RULE* **Split up the number from the right into groups of three digits; subtract in steps each group from the following one, and test the divisibility of the final remainder. If it is divisible, so is the original number.**

Examples

$$\begin{array}{rrrr} 53 \mid 081 & 72 \mid 853 & 13 \mid 890 \mid 981 & 921 \mid 752 \\ -\underline{53} & -\underline{72} & -\underline{13} \quad \underline{877} & -\underline{921} \\ 28 & 781 & 877 \quad 104 & -169 \end{array}$$

28 is divisible by 7   781 by 11   104 by 13   169 by 13,

and therefore the original numbers are.

For 11 you will recognise our rule of § 13. The final remainder can be negative (as -169).

Exercises Test (*a*) 60277, (*b*) 285692511 for divisibility by 7, 11, and 13. Compare also § 79.

The proof depends on the fact that 7 × 11 × 13 gives the 'fabulous number' 1001, whose Th and U are both 1. If I multiply it by any 1-, 2-, or 3-digit number, say 53, the series of digits is *repeated* in 3-groups.

$$1001 \times 5 = 5005; \quad 1001 \times 53 = 53053; \quad 1001 \times 890 = 890890.$$

Every multiple of 1001 is divisible by 7, 11, and 13.

If I subtract from	53081
the 1001-multiple 53053	53053
I have only the remainder	28

to test for divisibility. I can do this quickly according to the above rule. Its value lies in the fact that, instead of having to test a number with *more* than three digits, one has only to test a number with three digits.

An amusing story. Three collectors A, B, C came to a miser each with a request for a gift. 'Gladly', he said. 'Will each of you write down a three-digit number?' They wrote

A 345 B 687 C 908.

The miser then corrected himself: 'No, make a six-digit number by writing your number again'. So they wrote

A 345345 B 687687 C 908908.

'I will pay each of you the remainder which you get when Mr. A divides his number by 7, Mr. B divides his by 11, and Mr. C divides his by 13. Do you accept?'

Naturally the three accepted this mathematical sacrifice, and were overjoyed to find that at the best they would get 6 + 10 + 12 = $28. But alas, *not one* of them had a remainder. 'Comfort yourselves, gentlemen', consoled the miser, 'let Mr. A divide by 13, Mr. B by 7, and Mr. C by 11'. Now, they thought, something *must* happen—but it was exactly the same: there was *again* no remainder. 'Would you like to change your divisors once more?', asked the miser in a voice sweet as honey—but again nothing, nothing, nothing!

'Gentlemen, I sympathise with you. You chose the numbers yourselves, but fate has decided against you', and smiling he shut his door. Shaking their heads the three went down the stairs: how curious that fate should go against them three times! It would also have gone against them if they had divided by 77, 91, and 143. But we know that it was not fate, but the 1001-rule!

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§ 79 TWO GENERAL DIVISIBILITY RULES FOR ANY DIVISOR

such as 17, 19, 37, and so on can be found in methods *A* and *B*, which depend on the idea of *boundary-division* (§§ 80 ff.). They are not generally known. Normally if anyone wants to test whether 6385679 is divisible by 19, he has to divide by 19. The following simple, but rather curious, rules for divisibility avoid this work. I give both methods for those who are keen on numbers. The beginner should examine only the rules and the examples, and leave out the mathematical background.

(*A*) The method T + *h* U with the auxiliary number *h*

A I. **Rules for 7, 13, 17, 19.** A number is divisible

By 7, if its final remainder is of the form **T − 2U**.

The reader will naturally not understand this; he must learn the method by working through an example, then he will appreciate the concise form to which the detailed description is reduced, giving a rule which is easily remembered. The auxiliary number is negative here, h = −2.

Example Is 3206 divisible by 7?

(1) double the U: 2 × 6 = 12;
(2) subtract it from the T: 308;
(3) repeat (1) and (2): final remainder 14;
(4) conclusion: 14 is divisible by 7, *so* 3206 *is*.

$$\begin{array}{r} 3\,2\,0\,|\,6 \\ -1\,2 \\ \hline 3\,0|8 \\ 1\,6 \\ \hline 1\,4 \end{array}$$

By 13, if its final sum is of the form **T + 4U**.

By 17, if its final remainder is of the form **T − 5U**.

By 19, if its final sum is of the form **T + 2U**.

Examples

By 13, so T + 4U	By 17, so T − 5U	By 19, so T + 2U
1 1 0 8 | 9	1085943	89851
3 6	. . .579	. .87
1 1 4|4	. .812	912
1 6	1071	95
1 3 0, divisible by 13.	102	19, divisible by 19.
	0, divisible by 17.	

Exercises A Are (1) 142171 and (2) 10034147 divisible by 7, 13, 17, 19?

Hint. The auxiliary number h for any divisor which ends in 1, 3, 7, or 9 can be found below in *A II*, 1 and 2; it is necessary only for such numbers, since for even numbers (ending in 0, 2, 4, 6, or 8) and for those ending in 5 the simple divisibility rules of § 77 apply.

A II. **Proof of the method T + h U,** where h is the auxiliary number, by which the final digit U must be multiplied. The product is then added to the last digit but one, T, or subtracted from it. For divisors 7 and 17 the auxiliary numbers are -2 and -5, for divisors 13 and 19 they are $+4$ and $+2$ (*I* above).

(1) **For any divisor d an auxiliary number h can be given:**

Divisor with final digit	**1**	**3**	**7**	**9**
Auxiliary number $h=$	$-t$	$+(3t+1)$	$-(3t+2)$	$+(t+1)$

Here t denotes the tens digit; for 31, $t=3$.

Examples

Divisor d	Tens digit t	Auxiliary number h	Method T − h U
31	3	-3	T − 3U
73	7	$+[(3\times7)+1]=+22$	T + 22U
47	4	$-[(3\times4)+2]=-14$	T − 14U
59	5	$+6$	T + 6U
11	1	-1	T − U

The last example gives this 11-rule:

A number is divisible by 11 if its final remainder is of the form T − U. Do you recognise the old rule of § 13?

Examples Is 162245 divisible by 37 and 30806 by 73? The auxiliary number for 37 is given by $h=-11$, for 73 by $h=+22$. So the method goes

$$\begin{array}{l}
\text{T}-11\text{U}: \\
16224|5 \div 37 \\
\underline{-55} \qquad (-5\times11) \\
1616|9 \\
\underline{-99} \\
151|7 \\
\underline{-77} \\
74, \div 37 = 2;\ \text{it is divisible}
\end{array}$$

$$\begin{array}{l}
\text{T}+22\text{U}: \\
3080|6 \div 73 \\
\underline{+132} \qquad (6\times22) \\
321|2 \\
\underline{+44} \\
36|5 \\
\underline{+110} \\
146, \div 73 = 2;\ \text{it is divisible.}
\end{array}$$

(2) **How is this remarkable auxiliary number found?**

The general idea: instead of dividing by a 'difficult' divisor d (= 13) we divide by an 'easier' *ten-boundary* B $(=40)$. So we look for a multiple of 13 *which differs from the ten-boundary by* $+1$ *or* -1. So

$$(a)\ (3\times 13)+1=40 \qquad (b)\ (7\times 13)-1=90.$$

If, instead of dividing 1625 by 13, we divide it by 40 or 90, we must adjust for the error which we make in the division by adding in (*a*), subtracting in (*b*), the so-called complement-remainder R. We carry through this boundary-division (compare § 81):

$$\begin{array}{rl}
 & (a)\ \underline{1625}\div 40=41 \\
-(4\times 40) & 2 \\
+(1\times 4) & \underline{+4}\ \mathrm{R} \\
 & \underline{65} \\
-(1\times 40) & 25 \\
+(1\times 1) & \underline{+1}\ \mathrm{R} \\
 & 26,\ \div 13=2
\end{array}
\qquad
\begin{array}{r}
\text{Answer:} \\
3\times 41=123 \\
\underline{+2} \\
125
\end{array}
\qquad
\begin{array}{rl}
 & (b)\ \underline{1625}\div 90=17 \\
 & 72 \\
-(1\times 1) & \underline{-1}\ \mathrm{R} \\
 & \underline{715} \\
 & 85 \\
-(1\times 7) & \underline{-7}\ \mathrm{R} \\
 & 78,\ \div 13=6
\end{array}
\qquad
\begin{array}{r}
\text{Answer:} \\
7\times 17=119 \\
\underline{+6} \\
125
\end{array}$$

The answer of division (*a*) by 39 $(=3\times 13)$ is 41 remainder 26. For the 13-division it is 3 times larger, and since 2 comes from dividing the remainder by 13, we have $3\times 41=123$, $+2=125$. Similarly for (*b*). The answers are the same, of course.

The example of 13 shows that we could choose either of *two* ten-boundaries (40, 90), with an error of $+1$ or -1; we naturally choose the smaller.

For any divisor d ending in 3 (such as 43), $d=10t+3$ $(=(4\times 10)+3)$, where t $(=4)$ is the tens digit, the (smaller) ten-boundary B $(=130)$ is obtained by multiplying by 3:

$3(10t+3)=(3t\times 10)+9$, and adding 1 gives $B=(3t+1)\times 10=h\times 10$.

For $d=43$ we have $3d=3\times 43=129$, $+1=130=[(3\times 4)+1]\times 10=h\times 10$; the auxiliary number is $h=(3\times 4)+1=+13$, and the method is T+13U.

This discussion for $d=43$ can be written briefly:

$\dfrac{43\times 3 \mid +1}{13{\cdot}0}$ that is, $43\times 3=129$, $+1=130$, the boundary, and leaving off the zero gives the auxiliary number 13.

In the same way, for any divisor d you can quickly find the auxiliary number h, so for the examples above *II*, 1:

$$h=\frac{31 \mid -1}{-3{\cdot}0} \qquad \frac{73\times 3 \mid +1}{+22{\cdot}0} \qquad \frac{47\times 3 \mid -1}{-14{\cdot}0} \qquad \frac{59\times 1 \mid +1}{+6{\cdot}0}$$

Generally for a divisor ending in 7:

$$\frac{(10t+7)\times 3 \mid -1}{h=-(3t+2)}$$

Similarly the auxiliary numbers for divisors ending in 1, 3, and 9 can be formed (above II, 1).

(3) How is this remarkable method T + *h* U explained?

It is a division *from right to left*, which for example for a number *divisible* by 37 (or 13) must *go out*. We carry it through for the examples

(*a*) *A II*, 1 above

with negative auxiliary $h = -11$

$$\begin{array}{rl}
16224|5 \div 37 & \\
-55\ 5 & -5\times(3\times 37) \\ \hline
1616|90 & \\
-99\ 90 & -90\times(3\times 37) \\ \hline
151|700 & \\
-77\ 700 & -700\times(3\times 37) \\ \hline
74000 &
\end{array}$$

(*b*) *A I* above

with positive auxiliary $h = +4$:

$$\begin{array}{rl}
1108|9 \div 13 & \\
+36\ 0 & +(9\times 40)=9\times(3\times 13)+(9\times 1) \\ \hline
11449 & \\
-9 & -(9\times 1)\mathrm{R} \\ \hline
114|40 & \\
+16\ 00 & +(40\times 40)=40\times(3\times 13)+(40\times 1) \\ \hline
13040 & \\
-40 & -(40\times 1)\mathrm{R} \\ \hline
13000 &
\end{array}$$

(*a*) for negative auxiliary numbers the partial products are subtracted so $162245 - 795(3\times 37) = 74000$ or

$$162245 = 37(3\times 795) + 74000 = 37[(3\times 795) + 2000] = 37\times 4385:$$

(*b*) for a positive auxiliary number (+4), $49\,(3\times 13) = 13\times 147$ is added to 11089, giving $11089 + (13\times 147) = 13000$. Since 13000 is divisible by 13, so is $11089 = (13\times 1000) - (13\times 147) = 13\times 853$.

Note. Since testing for divisibility is here fundamentally a division from the right, in the case of a *divisor ending in 1*, as in $30388 \div 71 = 428$ (with $h = -7$), the answer to the division can be read off from the final digits in the development of T − 7U:

$$\begin{array}{ll}
3038|\mathbf{8} \div 71 & (-7) \\
298|\mathbf{2} & \\
28|\mathbf{4} & \\
0 &
\end{array}$$

According to *A II* (3*a*), 428 is taken away from 30388 71 times, giving remainder 0. If $30388 = m\times 71$, then $(m\times 71) - (428\times 71) = 0$, so $m = 428$.

For divisors ending in 3, 7, or 9, this is not immediately possible. I leave it to the reader to discover how to work out the answer from the final digits.

(*B*) The method of complement-remainders

***B I.* An example.** Is 38416 divisible by 7, $38416 \div (7)$? I write the divisor in brackets to show that the division is not to be carried out, but only a test for divisibility.

(1) Split up the number *from the right* into 2-groups: 3′84′16.

(2) Multiply the first group from the left (3) by 2(=6) and add the product to the second group (84+6=90).

(3) Repeat the process up to the end group (here 90×2=180, +16=196) and test for divisibility the *final remainder* (196) or 98, since you can again split 196 into 2-groups.

(4) If the final remainder is divisible, so is the original number.

$$\begin{array}{rr} & 3'84'16 \div (7) \\ 2\times 3 & \underline{6} \\ & 90 \\ & \underline{1\,80} \\ & 1'96 \\ & \underline{2} \\ & 98 \div 7 = 14 \\ & \text{divisible} \end{array}$$

This method reminds us of the 1001-rule (§ 78) by which we split into 3-groups and simply subtract the groups (not multiplied by 2) to the right. In fact both methods are essentially the same. This holds for *all* divisors; the methods differ only in their *grouping* and the *group-factor*. How can this be shown?

Proof of the method. Instead of dividing by 7, I divide by a unit-boundary, say 100 or 1000. Is 100 a multiple of 7? No, 100=(7×14)+**2**. If I wish to divide 384 by 7 and start by dividing by 100, so 384÷100=3 remainder 84, the answer for the 7-division is 14 times as large, 3×14=42, and the remainder is not 84, but 84 increased by the *complement-remainder* 3×2: 84+(3×2)=90. This is because, for each 100, the remainder is larger by 2. (Compare the more detailed work in § 81.) Only this final remainder 90 is to be tested for divisibility by 7. We set out the working like this:

$$\begin{array}{rl} & 384 \div (7) \\ & \underline{300} \\ & 84 \\ +(2\times 3) & \underline{6} \\ & 90 \end{array}$$

corresponding exactly to our method above: since it is only a question of divisibility, we disregard the answer completely.

$$\begin{array}{rl} & 3'84 \div (7) \\ +(2\times 3) & \underline{6} \\ & 90 \end{array}$$

B II. **Method for any divisor.**

(*a*) **The division-cross.** How do we recognise the groups and factors which belong to a given divisor? Through the '*division-cross*' (here for the divisor 7):

$$\frac{7\times 14 \mid +2}{100}$$

Below is the unit-boundary: as many *zeros* as there are *digits* in the group; on the right is the *group-factor*, here +2.

Examples (*a*) 96015 ÷ (111) (*b*) 425 ÷ (17)

$$\frac{111\times 9\mid +1}{1000}\qquad \begin{array}{rr} & 96'015\\ +(1\times 96) & \underline{+96}\\ & 111,\ \text{divisible}\end{array}\qquad\qquad \frac{6\times 17\mid -2}{100}\qquad \begin{array}{rr} & 4'25\\ -(2\times 4) & \underline{-8}\\ & 17,\ \text{divisible}\end{array}$$

Note on (*a*) We see that the '*adding*' 1001-rule (§ 78) tests for divisibility by 111.

(*b*) **Casting out multiples from a given number.** As in the 9-test we can cast out all 9-multiples (§ 6*b*), and similarly in, for example, 7-division test, all 7-multiples. We can do this at the start (*a*), or during the process (*b*):

$$\begin{array}{rr} (a) & 3'84'16\div(7)\\ -(7\times 12) & \underline{-84\ 14}\\ & 3\ 00\ 02\\ & \underline{6\qquad}\\ & 6\ \underline{12}\\ & 14,\ \text{divisible}\end{array}\qquad\qquad \frac{7\times 14\mid +2}{100}\qquad\qquad \begin{array}{r} (b)\ \ 3\ 84\ 16\div(7)\\ \underline{6\qquad\quad}\\ 90\qquad\quad\\ \underline{-84\qquad\quad}\\ 6\ \underline{12}\qquad\\ 28,\ \text{divisible}\end{array}$$

We are in no way bound by these 2-groups, so we can also subtract 7-multiples like this:

$$\begin{array}{r} 38'41'6\\ \underline{-35\ 35\ \ }\\ 3\ 06\ 6\ \text{ and then test:}\end{array}\qquad\qquad \begin{array}{r} 30'66\div(7)\\ \underline{60}\qquad\quad\\ 1|26\qquad\quad\\ \underline{2}\qquad\quad\\ 28,\ \text{divisible by 7.}\end{array}$$

We could also decrease the number 3066 by 7-multiples, from 3066 to 203 and finally to 63. In this attractive way the number crumbles in our hands until we know whether it is divisible or not.

(*c*) **A table of group-factors** gives us a quick check on those divisors *d* which have the same group-factor *f*. This is the remainder on dividing a unit-boundary *p* (*p* = 100 or 1000) by *d*. Example: *d* = 7; *p* = 100; 100 ÷ 7 = 14 remainder 2, so the group-factor is 2 for the divisor 7.

p	*f*	*d*					
100	1	3	9	11	33	99	
(2-groups)	2	7	14	49	98		
	−2	17	34	51	102		
	and so on.						
1000	1	27	37	111	333	999	
(3-groups)	−1	7	11	13	77	91	143
	and so on.						

Example Suppose you wish to test a number for divisibility by 77, so $d=77$; from the table you find that you must split off 3-groups and subtract each leading-group from its neighbour ($f=-1$). (This is, as you see, the 1001-rule of § 78!) Carry on the table for further group-factors.

For higher unit boundaries, 10000 and so on, it is not easy to test final remainders with 4 or more digits.

B III. Summary of the general rule.

(1) For the given divisor determine the unit-boundary and the complement-remainder by means of the division-cross;

(2) cast out multiples from the given number at the start and during the process ('divisor-packets out');

(3) split the number into 2- or 3-groups, according to the boundary 100 or 1000;

(4) multiply the groups by the remainder and collect together for the final remainder. If this is divisible, so is the original number.

Example 10034147 ÷ (19).

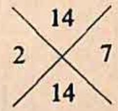
19-multiples out	Division-cross		Test	
10034147	$\dfrac{19\times 5 \mid +5}{100}$	11	20	24 ÷ (19)
515014			$\underline{55}$	$\underline{-5}$
131214			$\underline{75}$	19, divisible
112024		(−76)	−1	

Work the example without first casting out multiples.

Exercises B 1300068 ÷ (49), 30992 ÷ (13), 581217 ÷ (91), 1175 ÷ (47).

Application to the test. Just as we have previously formulated the 9-test with divisor 9 (§ 3), so we can now with our divisibility-rules formulate the 7-test and the 37-test. The method remains the same; we have only to determine the corresponding remainders instead of the 9-remainders. To test 63457 × 56728 = 3599788696 we carry through the 37-test, which is convenient because it is easy to cast out multiples ($3\times 37=111$, $6\times 37=222$, then 333...):

	63457	56728	3599788696	
$\dfrac{3\times 37\times 9 \mid +1}{1000}$	63013	50068	3044011030	The test holds
	$\underline{+63}$	$\underline{50}$	$\underline{47}$ $\underline{58}$	
	$\underline{76}$	$\underline{118}$	$\underline{58}$ $\underline{88}$	
Remainders:	2	7	14	



Although the power and simplicity of the old double test (9-11-test, § 15) can hardly be surpassed, the systematic extension, which now shows the 9-test to be a special case of a wider number-relation, gives great pleasure to anyone who is keen on numbers. He should certainly test the above example by means of the 7- or 17-test.

B IV. **Connection between the divisibility rules *A* and *B*.** If we compare method *B* with the previous method *A*, in which the auxiliary number *h* is present, we recognise its connection with *boundary-division*: in method *A* the smallest 10-boundary which differs from a multiple of the divisor plays an important part, in method *B* the *unit*-boundaries 100 and 1000 do.

We now devote our attention to this curious type of division.

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Boundary-division

§ 80 THE ESSENCE

of the method consists of replacing the divisor 29 in the sum $9143 \div 29$, for example, by its nearest boundary, here 30, and so in effect dividing by a 1-digit number (§ 73, note).

The method is suitable chiefly for division by 2-digit numbers, since generally division by a 3-digit number, say 387, leads to division by a 2-digit number, namely 39(0). This is a slight limitation on the method, which we can overcome, however. The division is particularly simple if the divisor (97) lies near a *unit-boundary* (100 or 1000). (See § 85.)

Two cases will be distinguished for practice.

(1) The divisor lies **below** the nearest boundary, such as 29, 38, 47; it must therefore be *complemented*: $29+1=30$; $47+3=50$. This is the most important case (§§ 81 ff.).

(2) It lies **above** the boundary, such as 52, 61, 84; it must therefore be *decreased* by the excess: $52-2=50$; $84-4=80$. (§ 84.)

First of all the method will be demonstrated by an example.

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§ 81 The case of a divisor below the boundary

Suppose we are given the sum 9143 ÷ 29. The divisor 29 being a prime number, we cannot use the neat factor-division of § 72. So we proceed like this: we divide by $29+1=30$; the first step of the division is $91 \div 30 = 3$ remainder 1.

However the right result is $91 \div 29 = 3$ remainder 4. The answer 3 agrees with the 30-division, but the *remainder* is *not* 1; this is 3 too small. Why? The number of times that 30 is contained in 90, that is 3, is the number by which the remainder is too small.

How do we adjust the error? Very simply: we add 3 to the remainder of the 30-division. This adjustment 3 we call the **complement-remainder** to distinguish it from the *division-remainder* 1.

$$\begin{array}{rl}
 & 91 \div 29 = 3 \\
3 \times 30 = & \underline{90} \\
\text{Division-remainder} & 1 \\
\text{Complement-remainder} & +\underline{3} \\
 & 4
\end{array}$$

Answer $91 \div 29 = 3$ remainder 4.

If you have understood the process up to now, you have grasped the point of boundary division. Isn't it simple? For the great advantage of dividing by a boundary, I pay with the complement-remainder. Then I have cancelled my debt and can confidently *repeat the process*:

(*a*) **Always divide first by the boundary,**
(*b*) **then adjust with the complement-remainder.**

Our example goes on:

$$\begin{array}{rl}
 & \underline{91}43 \div 29 = 315 \\
 & 1 \\
+(\mathbf{1} \times 3) & \underline{3} \\
 & \underline{44} \\
 & 14 \\
+(\mathbf{1} \times 1) & \underline{1} \\
 & \underline{153} \\
 & 3 \\
+(\mathbf{1} \times 5) & \underline{5} \\
 & 8
\end{array}$$

(*a*) Bring down the 4, $44 \div 30 = 1$ remainder 14.
(*b*) Complement-remainder **1** × 1: 15.

(*a*) Bring down the 3, $153 \div 30 = 5$ remainder 3.
(*b*) Complement-remainder $\mathbf{1} \times 5 = 5$: 8.

We have finished!
Answer: 315 remainder 8.

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§ 82 THE DIVISION CROSS

To bring order into the method we introduce the *division cross*, which for division by 29 looks like this:

$$\begin{array}{c} \underline{29 \mid +1} \\ 30 \end{array} \qquad \text{Say it as you write it down: '29+1=30'.}$$

29 is the *divisor*, +1 the *boundary complement* (or the *boundary distance*), 30 the *boundary*. This gives the

RULE **For boundary division:**

(*a*) **divide by the boundary (30);**
(*b*) **multiply the resulting answer (e.g. 5) by the boundary distance (1) and add the product (1 × 5) as the adjusting complement-remainder to the division-remainder.**

For the expert the whole rule is contained in the division cross:

$$\begin{array}{c} \underline{\textbf{divisor} \mid \textbf{+distance}} \\ \textbf{boundary} \end{array}$$

Examples

(*a*) $\underline{4387} \div 57 = 76$ rem. 55

$$\begin{array}{r} 18 \\ \underline{+21} \\ \underline{397} \\ 37 \\ \underline{+18} \\ 55 \end{array} \qquad \begin{array}{c} \underline{57 \mid +3} \\ 60 \end{array}$$

(*b*) $\underline{37} \div 29 = 1{\cdot}27\ldots$

$$\begin{array}{r} 7 \\ \underline{+1} \\ \underline{80} \\ 20 \\ \underline{+2} \\ 22 \end{array} \text{ and so on.} \qquad \begin{array}{c} \underline{29 \mid +1} \\ 30 \end{array}$$

(*c*) $\underline{63854} \div 297 = 214 \mid 296$

$$\begin{array}{r} 38 \\ \underline{+6} \\ \underline{445} \\ 145 \\ \underline{+3} \\ \underline{1484} \\ 284 \\ \underline{12} \\ 296 \end{array} \qquad \begin{array}{c} \underline{297 \mid +3} \\ 30 \end{array}$$

(*d*) $\underline{253872} \div 6989 = 36{\cdot}32\ldots$

$$\begin{array}{r} 4387 \\ \underline{33} \\ \underline{44202} \\ 2202 \\ \underline{66} \\ \underline{22680} \\ 1680 \\ \underline{33} \\ 1713 \end{array} \text{ and so on.} \qquad \begin{array}{c} \underline{6989 \mid +11} \\ 7000 \end{array}$$

In (*a*) with practice you can form the total remainder 39 from the division-remainder 18 and the complement-remainder 21 in your head.

In (*b*) you can, of course, express the answer as a decimal, by appending a zero to the remainder (8), thus overstepping the decimal point, and carrying on.

In (*c*) and (*d*) you can in both these sums see the effect of boundary division.

For divisors (below a boundary) which end in 5, 6, and 8, such as 65, 76, 88, factor division can always be used.

Exercises (*a*) 5465 ÷ 49, ÷ 37, ÷ 39, ÷ 97.
(*b*) 278452 ÷ 28, ÷ 66 (factor division as a test)
(*c*) as decimals to 4 places 5 ÷ 79, 62 ÷ 37
(*d*) 338573 ÷ 793, ÷ 3991.

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§ 83 SIMPLIFICATION FOR EXPERIENCED CALCULATORS

We choose our old example 9143 ÷ 29 of § 81. We have to divide by the boundary 30. However, this can be done easily (§ 72) by dividing first by 10—cut off the last digit—and then by 3. Instead of dividing by 29, we divide by 3, a surprising simplification!

$$\frac{29 \mid +1}{3(0)}$$

$\underline{914}(3) \div 29 = 315$

1	
3	(*a*) 9 ÷ 3 = 3 remainder 0;
4	(*b*) 1 down, complement-remainder 3 × (+1) = +3: 4.
14	(*a*) 4 ÷ 3 = 1 remainder 1;
1	(*b*) 4 down, complement-remainder 1 × (+1) = +1: 15.
15	
3	(*a*) 15 ÷ 3 = 5 remainder 0;
5	(*b*) 3 down, complement-remainder 5 × (+1) = +5: 8.
8	Answer 315 remainder 8.

In this surprisingly simple method there is just one mistake which one is liable to make: this consists of writing the complement-remainder under the division-remainder, that is as a T, instead of writing it under the digit *brought down*, that is as a U. If you avoid this carelessness and can work out the remainders in your head, then you can easily do divisions which would be quite troublesome by the old method.

Work through the following two sums, and you will learn to avoid the small stumbling-block which you may meet on the way!

(*a*) $\underline{954}(3) \div 29 = 32\underset{1}{8} = 329$ rem. 2

$$\begin{array}{r} \underline{8} \\ \underline{26} \\ 23 \\ \underline{+8} \\ \underline{\mathbf{31}} \\ 2 \end{array}$$

(*b*) $\underline{758}(2) \div 29 = 2\underset{11}{5} = 261$ rem. 13

$$\begin{array}{rr} & \underline{17} \\ & 28 \\ & \underline{+5} \\ & \underline{\mathbf{33}} \\ & 2 \\ \text{comp. rem.} & \underline{11} \\ & 13 \end{array} \qquad \frac{29 \mid +1}{3(0)}$$

In (*a*) in the course of the division we get 26 ÷ 3 = 8 remainder 2; 3 down: 23; complement-remainder 8: 31. Now this final remainder can be divided by 29. The resulting 1 is added to the 8. This happens because the addition of the complement-remainder 8 to the division-remainder 23 exceeds 29. Don't be bewildered by this but just divide; the partial answer (1) must be added to the previous one (8).

In (*b*) the same thing happens: the complement-remainder 5 raises the division-remainder 28 to 33, into which 3 goes **11** times. This 11 is written in the answer, as you see.

Don't think that this is a particularly tricky business; if you have understood the method properly, you would do it even if I had not made such a fuss about it!

Exercises in § 82.

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§ 84 THE CASE OF A NUMBER EXCEEDING THE BOUNDARY (§ 80 (2))

Suppose we have 4367 ÷ 51. Division cross $\frac{51 \mid -1}{50}$

Note The method is *exactly* the same as for a divisor below the boundary (§ 81), except that, as the division cross shows, the complement-remainder must be subtracted.

The method rarely gives any advantage, since for a divisor just above a boundary, such as 51, the division can easily be done by the usual method, and for *even* divisors (72, 24) factor-division can *always* be used. However, for completeness, the reader should work through the following

Exercises 35867 ÷ 11; ÷ 31; ÷ 43.

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§ 85 DIVISION BY A NUMBER NEAR A UNIT-BOUNDARY

Boundary division is surprisingly simple if the divisor is near a unit-boundary, such as 97 near 100, and 989 near 1000. We shall work through

Example (*a*) $46907 \div 97 = 483 \mid 56$ (*b*) $\overline{(4)6907 \div 97}$

$$\begin{array}{r} \underline{-400} \\ 69 \\ \underline{+12} \\ 810 \\ \underline{-800} \\ 10 \\ \underline{+24} \\ 347 \\ \underline{-300} \\ 47 \\ \underline{+9} \\ 56 \end{array} \qquad \begin{array}{c} \underline{97 \mid +3} \\ 100 \end{array} \qquad \begin{array}{r} 69 \\ \underline{12} \\ (8)10 \\ \underline{24} \\ (3)47 \\ \underline{9} \\ 56 \end{array}$$

In (*a*) if you work through according to the rule, you soon recognise two things:

(1) The partial answers 4, 8, 3 *agree* with the *leading digits* of the division sum, 469; 810; 347.

(2) The remainders 69; 10; 47 are then the digits *after* the leading digits.

So in the case of a unit-boundary you can *simplify* the work as in (*b*).

In (*b*) the leading digits are *cut off* (4), (8), (3) and read from top to bottom they give the answer 483. The 'beheaded' numbers 69, 10, 47 are the remainders.

Isn't this a wonderful method?

Further examples (*c*) 121984 ÷ 989; (*d*) 2313 ÷ 83; (*e*) 17 ÷ 93 to 2 places.

(*c*) $(1)21984 \div 989 = 123 \; 337$ (*d*) $(2)313 \div 83 = 27 \mid 72$

$$\begin{array}{r} \underline{11} \\ (2)308 \\ \underline{22} \\ (3)304 \\ \underline{33} \\ 337 \end{array} \qquad \begin{array}{c} \underline{989 \mid +11} \\ 1000 \end{array} \qquad \begin{array}{r} \underline{34} \\ (6)53 \\ \underline{102} \\ (1)55 \\ \underline{17} \\ 72 \end{array} \qquad \begin{array}{c} \underline{83 \mid +17} \\ 100 \end{array}$$

In (*d*) the line 653 gives first the quotient 6. However, at the next stage this is increased by the 1 from 155, which stands in the *same* column as the 6 and must therefore be *added* to the 6 to give the answer 27 (and not 261).

(*e*)

$$\begin{array}{r} 17 \div 93 = 0{\cdot}1827\ldots \\ \overline{(1)70} \\ 7 \\ \hline (7)70 \\ 49 \\ \hline (1)19 \\ 7 \\ \hline (2)60 \\ 14 \\ \hline (7)40 \end{array} \qquad \frac{93 \mid +7}{100}$$

In (*e*) of course the answer can always be given as a *decimal*, rather than with a remainder, by appending a zero to each remainder.

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Exercises 4481 ÷ 93; 668345 ÷ 994; 16·5 ÷ 9·7 (to 3 places).

Note One can work in the same way with divisors *above* a unit-boundary, for example 4481 ÷ 105. I leave this to the reader; compare § 84. Much more unpleasant than these divisors are those *below* the boundary.

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§ 86 OTHER KINDS OF DIVISION BY NUMBERS NEAR A UNIT-BOUNDARY

We take the first example of § 85, namely 46907 ÷ 97, which is naturally done by dividing by the boundary 100. Up to now we have added on the complement-remainder when it becomes due; now we divide the *whole* number by 100 and determine the *total remainder*. An example will make this clear.

$$\begin{array}{r} (469)07 \div 97 \\ 14\ 07 \\ \hline (14)14 \\ 42 \\ \hline 56 \end{array} \qquad \frac{97 \mid +3}{100} \qquad\qquad \begin{array}{r} (121)984 \div 989 = 123 \mid 337 \\ 1\ 331 \\ \hline (2)315 \\ 22 \\ \hline 337 \end{array} \qquad \frac{989 \mid +11}{1000}$$

The method

(1) A number is divided by 100 by cutting off two digits, 469 | 07. Answer: 469 remainder 07.

(2) Complement-remainder, 3 × 469 = 1407. Total remainder 1414. This again can be divided by 100.

(3) 14 | 14. Calculation of remainder as before.

(4) Final answer 469 + 14 = 483, remainder 56.

The second example 121984 ÷ 989 we have already done in § 85. Now you can choose which method you prefer. Both are attractive, and, if the idea of complementing the remainder has been understood, they are so easy that they can be offered in school as delicacies in addition to the ordinary diet of division.

Exercises and *Examples* in § 85.

For the experienced calculator.

6387 ÷ 33; if a **multiple** of the divisor (33) is near a **unit-boundary**, $3 \times 33 \simeq 100$, you can work like this:

6387 ÷ 33; here the division cross is
Say and write: '33 times 3 + 1 = 100'.

$$\begin{array}{r|l} 33 & +1 \\ \hline 100 & \times 3 \end{array}$$

What does this mean? 33 is contained 3 times in 100; 2 × 3 times in 200; H × 3 in H × 100. This means then: **multiply the H by 3.** But surely you have made a mistake, you have forgotten the *remainder*! What is the remainder on dividing by 33? With each 100 there is the remainder 1; with H × 100 therefore H × 1. So the division cross says

(*a*) Divide by 100, therefore cut off 2 digits: 63 | 87.
(*b*) Multiply the H by 3: 63 × 3 = 189
(*c*) Add to the digits cut off, 87, the complement-remainder 1 × 63 = 63: so 63 + 87 = 150.
(*d*) Divide this remainder 150 again; either straight away by 33, if it is fairly *small*, thus 150 ÷ 33 = 4 remainder 18; or apply the process (*a*) to (*d*) *again*: 1 | 50; 3 remainder 50 + 1. 51 ÷ 33 = 1 remainder 18.
(*e*) Final answer: 193 remainder 18.

$$\begin{array}{r|l} 63 & 87 \div 33 \\ \hline 189 & 63 \\ \hline 3 & 150 \\ 1 & 51 \\ \hline 193 & 18 \end{array}$$

In this surprising way a whole series of *divisions* can be carried out by *multiplying* by 1-digit numbers. Essentially this is what happens in division by 5, 25, and 125 (§§ 67 to 69): since the T is multiplied by 2, the H by 4, and the Th by 8 respectively, the unit boundary is multiplied by that factor which brings the divisor into its 'neighbourhood'; here the multiple coincides with it exactly. These three divisors are therefore proper special cases of the *general* case given here. The experienced calculator can work out smaller sums in his head, for example

$407 \div 33 \qquad \frac{33 \mid +1}{100 \mid \times 3} \qquad \text{so } 4 \times 3 = 12 \mid 7 + (1 \times 4) = 11$

and $435 \div 33 = 13 \mid 6$; $817 \div 49 = 16 \mid 33$; $538 \div 97 = 5 \mid 53$;
$5337 \div 333 = 15 + 1 \mid 4 + 5$, so $16 \mid 9$; $724 \div 89 = 8 \mid 12$.

Of course, many sums can also be done easily by factor-division, which the reader will generally prefer to boundary-division, since, apart from the factorisation, the work demands less concentration.

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§87 THE COMBINATION OF FACTOR- AND BOUNDARY-DIVISION

is, for skilled calculators, the summit of all work on division. By means of them most divisions can be done by a pleasant short cut, whereas the schoolboy usually has to keep to the main road.

(*a*) $265838 \mid 64 \div 725 = 36667$ remainder 289

$725 = 25 \times 29$

÷ 25 $106335(4) \mid 14$

19 $\mid \div 29 = 36667 \mid 11$

19

19

21

11 $\quad 11 \times 25 = 275, \ +14 = 289$

$\frac{29 \mid +1}{3(0)}$

(*b*) $3673757 \div 7896 \ (= 8 \times 987) = 465$ remainder 2117

÷ 8 $(4)59219 \mid 5 \div 987$

52

(6)441

(5)199

264 $\quad (\times 8 = 2112, \ +5 = 2117)$

$\frac{987 \mid +13}{1000}$

Exercises (*a*) $483764 \div 355$; (*b*) $351181 \div 1368$; (*c*) $30313233 \div 7864$.

Application: factor-division by a prime number.

Suppose we are given 5386÷43 (=125 | 11). The divisor is a prime number, which means that we cannot use factor-division.

Now we consider the matter thus: every prime number (except 2) is odd; so its left and right neighbours (42 and 44) are even, and therefore at least divisible by 2. It is usual to choose the one which factorises best, but in our case there is little to choose: 42=6×7 and 44=4×11. For the divisor 37 it would be best to choose 36=6×6; the other one, 38=2×19, has less convenient factors.

Our method is simply:

Instead of dividing by the number itself, 'factor-divide' by one of its neighbours.

With this hint for the general solution I leave the experienced calculator to find his own way in particular cases.

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§ 88 RETROSPECT

Here we come to the end of our journey through the territory of division. We can look back on *division by particular numbers*, *factor-division*, and *boundary-division*: what a wealth of pleasant roads, lanes, and footpaths cross the land of division, making it an attractive and exciting expedition for the adventurous calculator. Perhaps schools could undertake such an expedition for pleasure and profit!

At the same time we have reached the end of our wanderings through the four kinds of calculation. Even if you have come only a little way with me, you will not only have gained some useful hints, but also, I hope, when you work with numbers you will be calmer, less anxious, and probably more cheerful. Certainly numbers can be dreary, but they can always disclose some of their magic, even in the simplest calculations.

III ESTIMATES

§ 89 AN ESTIMATE DETERMINES THE ORDER OF MAGNITUDE

At the end of the four types of calculation we have learnt that an *estimate* is a safety precaution which should be carried out before starting work, because it limits the region in which the answer can lie. It should never be forgotten, particularly by inexperienced calculators, for it gives the feeling of certainty, which is an incentive which cannot be overestimated. An estimate also provides a judgment about the answer. Teachers know that even older boys are liable to give the price of 10 lb of flour as $150 (instead of $1·50). So it is understandable that a less striking impossibility is even less likely to be noticed. Therefore in school everybody should be given the short, but so important

RULE **Estimate before, test afterwards.**

What does it mean to 'estimate' a sum 98×32? It means that instead of working with the given numbers we work with 'easier' neighbouring round numbers, which do not give the right answer (3136), but more quickly and surely a value (3000) in its neighbourhood.

$$98 \times 32 \simeq 100 \times 30 = 3000; \quad \text{exactly, } 3136.$$

Yes, but what is the use of an estimate, if it doesn't guarantee the right answer? It determines the *order of magnitude* of the answer, and therefore a neighbouring value, which tells us with certainty whether the answer is of the order H Th or Th or U. This is all one needs to know in many cases in everyday life (compare §§ 1, 2, and 57).

Estimates can be coarse or fine: coarse values are provided by *boundary estimates* (§ 90), the finer ones by methods already learnt (see § 91) or by *stem-fractions* (§ 92).

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§ 90 BOUNDARY ESTIMATES

are quite sufficient to determine the order of magnitude:

RULE **Replace both numbers by neighbouring boundaries (round numbers).**

$$\begin{array}{rr} 327 \times 687 & 62735 \div 452 \\ \simeq 300 \times 700 = 210000 & \simeq 60000 \div 400 = 150 \\ \text{exactly, } 224649 & \text{almost exactly, } 139 \end{array}$$

Note For *multiplication* choose round numbers in *opposite* directions, one above, the other below; for *division* in the *same* direction, both above or both below.

For example, if for the division we had said $60000 \div 500$, we should have obtained a worse estimate 120; though this would, of course, still be quite sufficient for the order of magnitude.

The beginner is often very nervous of rounding off numbers. It does certainly need some skill, but not much. I hope he will overcome this nervousness by going through our examples.

For decimals

estimates are particularly important. Here one can test the order of magnitude by mastering the **point rule.**

(*a*) **The rule for multiplication is: Multiply without the points and then cut off as many digits as there are in the two factors together after the point (here 4).**

$$\begin{array}{r} 7{\cdot}37 \times 0{\cdot}25 \\ \underline{737 \times 25} \\ \underline{18425} \\ 1{\cdot}8425 \end{array}$$

Estimate: $7 \times \frac{3}{10} = 2{\cdot}1$; or $7 \times \frac{1}{4} = 1{\cdot}75$; or $7 \times 0{\cdot}3 \simeq 2$.

In mental work, say the solution of the sum $0{\cdot}8 \times 0{\cdot}03$ like this: '$8 \times 3 = 24$, with 3 (decimal) places, so $0{\cdot}024$'. Or '$0{\cdot}09^2 = 81$ with 4 places, so $0{\cdot}0081$'.

(*b*) **The point rule for division.** Distinguish two cases for the divisor: (1) a whole number, (2) a decimal.

(1) **Decimal divided by a whole number,** such as $1{\cdot}5 \div 3$.

RULE *Place the point where you step over it in division.*

$1 \div 3 = 0$—point—$15 \div 3 = 5$; answer $1{\cdot}5 \div 3 = 0{\cdot}5$.

(2) **The divisor is a decimal:**

$$\begin{array}{rlr} 15 \div 0{\cdot}3 = & \text{enlarged:} & 150 \div 3 = 50 \\ 0{\cdot}15 \div 0{\cdot}03 = & & 15 \div 3 = \ \ 5 \end{array}$$

RULE *Enlarge the sum so that the divisor is a whole number.* To enlarge: shift the point in both numbers by the *same* number of places to the right.

The principle is this. We can only divide by a *whole number*. If the divisor is a decimal, make it a whole number by enlarging it. For an estimate carry through the enlargement and then round off:

$$74{\cdot}8 \div 3{\cdot}5 = 748 \div 35 \simeq 700 \div 30 \simeq 20 \text{ (exactly, } 21{\cdot}37)$$
$$4{\cdot}8732 \div 0{\cdot}075 = 4873{\cdot}2 \div 75 \simeq 5000 \div 80 = 500 \div 8 \simeq 60 \text{ (exactly, } 65).$$

This kind of estimate replaces the given numbers by boundaries, which often bring in an unpleasantly large number of zeros:

$$17895 \times 93017 \simeq 18000 \times 90000.$$

These must be handled carefully like raw eggs, or you may drop one or two! We rule out this uncertainty by writing

Round numbers in powers of 10

It is quite simple:

$$100 = 10 \times 10 = 10^2; \quad 1000 = 10 \times 10 \times 10 = 10^3; \quad \ldots; \quad 100000 = 10^5.$$

All unit-boundaries, and only those, are powers of 10. The *index* always gives the *number of zeros*. From another boundary, e.g. 600, we cut off the unit-boundary and write

$$600 = 6 \times 100 = 6 \times 10^2; \quad 60 = 6 \times 10^1; \quad 6000000 = 6 \times 10^6.$$

Now we need no longer have zeros all over the place:

$$18000 \times 90000 = 18 \times 10^3 \times 9 \times 10^4 = 18 \times 9 \times 10^3 \times 10^4 = 162 \times 10^7,$$

and therefore 162 with 7 zeros, or in round figures 16 with 8 zeros. We need only remember the simple

POWER RULE **Add the indices for multiplication, subtract for division.**

Examples (see the beginning of this Section)

$$3 \times 10^2 \times 7 \times 10^2 = 21 \times 10^{2+2} = 21 \times 10^4 = 210000;$$
$$(60 \times 10^3) \div (4 \times 10^2) = (60 \div 4) \times 10^{3-2} = 150.$$

Negative indices are dealt with like this:

$$62 \div 308 \simeq (6 \div 3) \times 10^{1-2} = 2 \times 10^{-1} = 0{\cdot}2.$$

Note *If the index is negative put the corresponding number of zeros before the first significant digit.*

The power rule is particularly important for work with decimals, since it includes the *point rule*, though for division one must be able to work with directed numbers.

$$8{\cdot}4 \div 0{\cdot}02 = (84 \div 2) \times 10^{-1-(-2)} = 42 \times 10^{-1+2} = 42 \times 10^1 = 420.$$

One learns the significance of estimation with powers first in *chains of fractions*, like those which a physicist is always meeting.

$$\frac{0{\cdot}062\times760\times291}{32{\cdot}65\times733{\cdot}22\times273\times0{\cdot}001293}\simeq\frac{6\times10^{-2}\times7\times10^{2}\times3\times10^{2}}{3\times10^{1}\times7\times10^{2}\times3\times10^{2}\times10^{-3}}$$

$$=\frac{6\times7\times3\times10^{2}}{3\times7\times3\times10^{2}}=\frac{6}{3}=2.$$

The order of magnitude of this difficult-looking expression—it is a calculation of the density of steam—is therefore 2, not 20 or 0·2; the exact value 1·6 is quite well approximated by the coarse reduction. The numerical value of such fractions can be obtained with the slide rule, but such work is unthinkable without estimation.

The teacher is recommended to introduce unit-boundaries as powers of 10 quite early, in the fifth or sixth year of arithmetic, and I would remind him of what I said in § 61 (*A*) about writing in powers. The young calculator will then become more confident with them, he will profit by their conciseness, and experience them not as something special and highly mathematical, as usually happens with the 'official' introduction of work with powers. And by the seventh year he may understand the rule of § 66 for decimals, which is not very easy to grasp otherwise: 'to divide a number by 10^x, move the decimal point x places to the left'.

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§ 91 ESTIMATION WITH TRICKS

General rules cannot be given for this; one can estimate by means of neighbouring numbers, which allow a known device to be used, for example

25-trick (§ 30)	5-square (§ 62 (*A*))	(§ 49 (1))
3275×245	$345\times0{\cdot}35$	$6{\cdot}3\times6{\cdot}8$
$\simeq32\times25\times10^{2+1}$	$\simeq35^2\times10^{1-2}$	$\simeq6{\cdot}3\times6{\cdot}7=42{\cdot}21$
$=8\times10^5$,	$=1225\times10^{-1}=122{\cdot}5$	$\simeq6{\cdot}2\times6{\cdot}8=42{\cdot}16$
exactly, 802375	exactly, 120·75	exactly, 42·84

The second example can also be estimated very easily like this:

$$345\times0{\cdot}35\simeq345\times\tfrac{1}{3}=115,\text{ that is, by}$$

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§ 92 ESTIMATION BY STEM-FRACTIONS

A stem-fraction has numerator 1, for example $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{7}$. A 100-stem-fraction has numerator 100, such as $\frac{100}{3}$, $\frac{100}{7}$.

With these 100-stem-fractions quite a tight chain of numbers can be formed, whose intermediate numbers can be approximated easily, for example

$$42\times 33 \simeq 42\times\frac{100}{3}=\frac{42}{3}\times 100=1400.$$

The most important 100-stem-fractions

$45-\mathbf{50}-55\simeq\frac{1}{2}\times 100$
$30-\mathbf{33}-35\quad\frac{1}{3}$
$22-\mathbf{25}-28\quad\frac{1}{4}$
$18-\mathbf{20}-22\quad\frac{1}{5}$
$16-\mathbf{17}-18\quad\frac{1}{6}$
$13-\mathbf{14}-15\quad\frac{1}{7}$
$\mathbf{12}\quad\frac{1}{8}$
$\mathbf{11}\quad\frac{1}{9}$

The best approximations are printed in heavy type. For multiples of 50 downwards there is an increasingly tightly-drawn net of good approximations.

Examples:

$68\times 14\simeq 70\times 14\simeq\frac{70}{7}\times 100=1000.$

$5327\times 19\simeq\frac{5327}{5}\times 100\simeq 106500.$

The experienced calculator can also use the

1000-stem-fractions

$111\simeq\frac{1}{9}\times 1000$
$125\quad\frac{1}{8}$
$143\quad\frac{1}{7}$
$166\quad\frac{1}{6}$
$200\quad\frac{1}{5}$
$250\quad\frac{1}{4}$
$333\quad\frac{1}{3}$
$500\quad\frac{1}{2}$

$$72\times 185\simeq\frac{72}{5}\times 1000\simeq 14000 \text{ or}$$
$$\simeq\frac{72}{6}\times 1000=12000.$$

Here the exact value is enclosed between limits; we can write for short:

$$12000<72\times 185<14000$$

and read: '72×185 is greater than 12000 and less than 14000'. The right value is 13320.

Exercises The reader should find an estimate for all solutions; these few just serve as introductory examples.

(1) Estimation by boundaries or with tricks:

$227\times 185,\quad 4685\times 27,\quad 317\times 127,\quad 445\times 17,\quad 2675\times 8277.$

(2) Estimation by stem-fractions:

(*a*) $36\times 17,\quad 66\times 16,\quad 56\times 12,\quad 74\times 19,\quad 434\times 13.$

(*b*) $315\times 32,\quad 16425\times 21,\quad 48\times 122,\quad 672\times 335,\quad 12035\times 189.$

In every case find the limits between which the answer must lie. Then you will soon acquire that skill in estimation which is so important in everyday work. You will learn this particularly in percentages, to which we now turn.

IV PERCENTAGES

§ 93 THE KEY EQUATION

Everybody knows that many quite good calculators are incredibly scared of the word 'percent'. In schools one often meets quite advanced classes who can differentiate and integrate, but who are unable to give immediately, either exactly or approximately, the percentage if 4 of a class of 28 stay down.

It is usually set out like this:

there are 4 pupils out of 28,
x pupils out of 100,

and then the rule of three is laboriously applied. One can then say immediately that it is about 14%, or exactly $14\frac{2}{7}\%$. But it is child's play if one knows what 'percent' really means.

If someone is asked for 3% of $6000, the answer is usually begun, quite correctly but unsuitably, '3% means $3 in every $100'. It is much better to say that 3% is just $\frac{3}{100}$ of it. So we remember

the key equation for percentages:

1% of a number is just $\frac{1}{100}$ of it.

It follows that 2% is $\frac{2}{100}$; 3% is $\frac{3}{100}$. In decimals: $2\% = 0{\cdot}02$.

Generally $p\%$ **of a number is** $\frac{p}{100}$ **of it.**

Remember—a percentage is just a fraction with denominator 100.

* *

§ 94 THE IDEA OF PERCENTAGES

The number whose percentage is to be found is called the *original value*, for instance $6000 and 28 pupils in the examples of § 93.

3% is the *percentage-ratio* (p-ratio). 3% of \$6000 is $6000 \times \frac{3}{100} = 60 \times 3 =$ \$180. This value \$180 is the *percentage-value* (p-value).

So we distinguish between the original value, p-value, and **p-ratio** in our

school example	28	4	$14\frac{2}{7}$%;
money example	6000	180	3%.

Three fundamental types of sum occur in percentages.

(1) **The p-value** is required when the original value and p-ratio are given (§ 95). Example: what is 3% of \$6000?

(2) **The p-ratio** is required when the original and p-values are given (§§ 96, 97). Example: what % is \$180 of \$6000?

(3) **The original value** is required when the p-ratio and p-value are given (§ 98). Example: What is the capital in \$ if \$180 is exactly 3%?

I do not propose to give an introduction to percentages, but just show how to solve quickly the first two types of sum, which are the ones which occur most frequently in everyday life.

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The fundamental types of sum

The first type: calculation of the p-value

§ 95 THE P-VALUE CAN EASILY BE FOUND BY THE

RULE **Cut off two places from the original value** (by a decimal point or a stroke) **and multiply by the p-ratio.**

7% of 684 is 6·84 × 7 = 47·88; shorter, 7% of 6′84 = 47·88.
8% of 18 is 0·18 × 8 = 1·44; shorter, 8% of ′18 = 1·44.

The proof follows from the key equation of § 93. The mathematician learns the rule in the **formula for percentages**:

$$v = g \times \frac{p}{100} = \frac{g}{100} \times p,$$

where v is the p-value, g the original value and p the p-ratio.

Since p% is just $p/100$ of the original value, there is a whole series of p-ratios which can be expressed as stem-fractions of the original value; these can often be used for working out the exact value, and always for an estimate.

1% is $\frac{1}{100}$ of the original value.
10% is $\frac{1}{10}$,, ,,
20% is $\frac{1}{5}$,, ,,
25% is $\frac{1}{4}$,, ,,
$33\frac{1}{3}$% is $\frac{1}{3}$,, ,,
50% is $\frac{1}{2}$,, ,,

Also:
100% is the original value $\frac{100}{100}=1$,
200% is double the original value,
300% is three times the original value,
and so on
$n \times 100$% is n times the original value.

Everybody must know these p-ratios as stem-fractions.

Examples 25% of \$6832 is \$6832 ÷ 4 = \$1708
200% of \$75 is \$75 × 2 = \$150
1% of \$5 is \$5 ÷ 100 = 5 cents.

Remember 1% of a number of \$ is the *same* number of cents.

For the experienced calculator. By means of the summary of stem-fractions and percentage ratios of § 97(*b*) he can, if he reads from right to left, obtain an even tighter net of intermediate values by means of which he can quickly work out a percentage, at least approximately.

Example What is 17% of 36? A little more than 6! Why? Since $16\frac{2}{3}$% is exactly $\frac{1}{6}$ of the original value. Or 12% of \$32493? A little less than \$4060, since $12\frac{1}{2}$% is $\frac{1}{8}$ of it.

Splitting up the p-ratio. If the p-ratio is a mixed number such as $3\frac{1}{2}$% or $5\frac{7}{8}$%, it is often easier to work by splitting up the p-ratio than by the rule.

$3\frac{1}{2}$% of \$618	
$\frac{1}{2}$	3·09
3	18·54
$3\frac{1}{2}$%	\$21·63

$5\frac{7}{8}$% of \$6′18	
6	37·08
$-\frac{1}{8}$	0·77
$5\frac{7}{8}$	\$36·31

18% of \$618	
20	123·60
−2	12·36
18%	\$111·24

Notes

(1) Clever splitting up can save a lot of work and make the calculation very attractive.

(2) Always start with 1%; I remember this by putting a stroke in the original value, as in the second example, 6′18.

(3) Splitting up can of course be used to advantage with integral p-ratios such as 18% = 20% − 2% or 26% = 1% + 25% (25-trick!). Here as always the main rule holds: keep your eyes open!

Exercises Work out approximately and exactly the following percentages:

(*a*) 11% of 879; 25% of \$2345·50; 8% of 48573;

(*b*) 45% of 4500; 34% of \$378·75; 120% of \$4592.

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The second type: calculation of the p-ratio

§ 96 THE P-RATIO

If 5 of a class of 25 children are away, the 'ratio' is 5:25 (say '5 to 25'). A ratio is best written as a fraction $\frac{5}{25}$, since it is easier to work with in that form. From such a fraction we can read off the ratio. If instead of the real 'original value' 25 we choose the *comparison-number* 100, we get the 100-ratio, or p-ratio; here $\frac{5}{25}=\frac{20}{100}$, so 20%.

One further example. If a road rises 30 ft in a horizontal distance of 500 ft, what is the slope? As a ratio 30/500, as a percentage $\frac{6}{100}=6\%$.

Why do we choose such a comparison-number? The answer lies in the name: in business, technical, and public life, numbers continually have to be compared. If 12 of another class of 61 children are away, is this more or less than in the previous class, where the numbers were 5 and 25? In *actual numbers* more are away, 12 instead of 5; but compared with the total, *as a ratio*, less are away, because $\frac{12}{61}$ is smaller than $\frac{5}{25}$. But it is not so easy to compare two fractions at a glance as two *whole numbers*. If we give them the same denominator, however, we only have to compare their numerators, two *whole numbers*; and 100 is a natural choice for common denominator.

Which fraction is larger, $\frac{7}{12}$ or $\frac{11}{18}$? This is difficult to tell without further work. We bring them both to the same denominator 36, giving $\frac{21}{36}$ and $\frac{22}{36}$. So $\frac{11}{18}$ is larger than $\frac{7}{12}$ by $\frac{1}{36}$. In our example above of $\frac{5}{25}$ and $\frac{12}{61}$ we can proceed like this: $\frac{5}{25}=\frac{1}{5}$; $\frac{12}{60}=\frac{1}{5}$, but $\frac{12}{61}$ is smaller than $\frac{12}{60}$, and so smaller than $\frac{5}{25}$. Both these examples make it understandable why, in order to compare two ratios, a fixed comparison-number such as 100 should be chosen.

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§ 97 WORKING OUT THE P-RATIO

If the proportion of 5 children ill out of 25 is to be expressed as a percentage, first form the fraction $\frac{5}{25}$. For working out the percentage, remember the

MAIN RULE **First form the fraction from the p-value and original value.** For working out the p-ratio there are **three** (short) **ways**:

(*a*) enlarging the denominator to 100,

(*b*) contracting to a stem-fraction,

(*c*) working in decimals.

Example

(*a*) $\frac{4}{20}$, enlarged by 5, $=\frac{20}{100}=20\%$;

(*b*) $\frac{4}{20}$, contracted by 4, $=\frac{1}{5}=20\%$;

(*c*) $\frac{4}{20}$, converted to a decimal, $=4\div 20=0{\cdot}20=20\%$.

Methods (*a*) and (*b*) are not always possible (but are still good for approximation), but method (*c*) is always possible: it is, so to speak, the 'main road'.

(*a*) *RULE* **Enlarge (or contract) the fraction to denominator 100.**

A fraction is said to be enlarged if numerator and denominator are multiplied by the same number; contracted if they are divided by the same number.

Examples $\frac{5}{25}$, enlarged by 4, $\frac{20}{100}$, so 20%;
$\frac{3}{200}$, contracted by 2, $\frac{1{\cdot}5}{100}$, so 1·5%;

$\frac{3}{17}$; there is no simple number which will bring the denominator 17 to 100, since 17 is not a factor of 100. The number 6 nearly does it:

$\frac{3}{17}$, enlarged by 6, $\frac{18}{102}$, so roughly 18%.

(*b*) *RULE* **Contract the fraction to a stem-fraction (or a multiple of one).** If this is impossible, choose a neighbouring number.

Examples $\frac{5}{25}=\frac{1}{5}=20\%$; $\frac{7}{63}=\frac{1}{9}=11\frac{1}{9}\%$.

Remember the table of **stem-fractions—p-ratios:**

The original value	100%	Also	$\frac{1}{11}$ is approx.	9%
$\frac{1}{2}$ of the original value	50%		$\frac{1}{12}$ is	$8\frac{1}{3}\%$
$\frac{1}{3}$,, ,,	$33\frac{1}{3}\%$		$\frac{1}{14}$ is approx.	7%
$\frac{1}{4}$,, ,,	25%		$\frac{2}{3}$ is approx.	67%
$\frac{1}{5}$,, ,,	20%		$\frac{3}{4}$ is	75%
$\frac{1}{6}$,, ,,	$16\frac{2}{3}\%$		and	
$\frac{1}{7}$,, ,,	$14\frac{2}{7}\%$		2 is	200%
$\frac{1}{8}$,, ,,	$12\frac{1}{2}\%$		$2\frac{1}{2}$ is	250%
$\frac{1}{9}$,, ,,	$11\frac{1}{9}\%$		3 is	300%
$\frac{1}{10}$,, ,,	10%		and so on.	

Examples (1) **Stem-fractions**

4 children out of 28 children	$\frac{4}{28}=\frac{1}{7}=14\frac{2}{7}\%\simeq 14\%$.
7 articles ,, ,, 42 articles	$\frac{7}{42}=\frac{1}{6}=16\frac{2}{3}\%\simeq 17\%$.
\$14 out of \$126	$\frac{14}{126}=\frac{1}{9}=11\frac{1}{9}\%\simeq 11\%$.
\$45 ,, ,, \$495	$\frac{45}{495}=\frac{1}{11}\simeq 9\%$.

(2) Multiples of stem-fractions

27 articles out of 63 articles $\frac{27}{63}=\frac{3}{7}=3\times\frac{1}{7}=3\times 14\tfrac{2}{7}\%\simeq 43\%$.
35 ,, ,, 56 ,, $\frac{35}{56}=\frac{5}{8}=5\times 12\tfrac{1}{2}\%=62\tfrac{1}{2}\%$.
65 ,, ,, 78 ,, $\frac{65}{78}=\frac{5}{6}=5\times 16\tfrac{2}{3}\%=83\tfrac{1}{3}\%$.
\$70 to \$40 $\frac{70}{40}=\frac{7}{4}=175\%$.

(3) Approximations

18 out of 28 $\frac{18}{28}=\frac{9}{14}\simeq 64\%$;
23 ,, 86 $\frac{23}{86}\simeq\frac{1}{4}=25\%$;
17 ,, 54 $\frac{17}{54}<\frac{18}{54}=\frac{1}{3}=33\tfrac{1}{3}\%$; (< means ‘less than’)
18 ,, 390 $\frac{18}{390}<\frac{20}{400}=\frac{1}{20}=5\%$; exactly, 4·6%;
3652 ,, 48476 $\frac{3652}{48476}\simeq\frac{36}{484}\simeq\frac{3}{40}=3\times 2\tfrac{1}{2}\%=7\tfrac{1}{2}\%$; exactly, 7·53%.

When you have worked through these examples, you will understand what an advantage it is to know the p-ratios as stem-fractions of the original value. You can then easily work out the nearest p-ratio. In most cases it does not come to an exact value; if 3652 out of 48476 inhabitants are out of work, it is sufficient to know that this is about 7%; certainly not 4% or 12%; the exact percentage is 7·5%.

(*c*) *RULE* **Work out the fraction as a decimal to two places, and this gives the p-ratio.**

$$\tfrac{5}{25}=5\div 25=0{\cdot}20=20\%;\quad \tfrac{7}{63}=7\div 63=0{\cdot}111=11{\cdot}1\%.$$

Why? The first two digits after the decimal point are the hundredths, and so according to the key equation of § 93,

$$0{\cdot}20=\tfrac{20}{100}=20\%.$$

This method *always* leads to the answer. Work to three places and round off. If the work is hard because of a complicated denominator, remember the factor-division of § 73. Example: if 7 out of 15 vineyards are attacked by disease, what percentage is this?

Fraction $\frac{7}{15}$; $7\div 15$ (or 3×5); answer 47%.
$\div 5$ 1·4
$\div 3$ 0·466

Hint. The slide rule always expresses a fraction as a decimal, so you can use it to work out p-ratios.

Note Anybody who wants to be an expert calculator must master these three methods (*a*), (*b*), (*c*). You never need the rule of three.

Exercises Work out the p-ratios of (*a*) $\frac{47}{94}$, $\frac{52}{156}$, $\frac{7}{12}$, $\frac{3}{7}$; (*b*) $\frac{75}{450}$, $\frac{18}{27}$, $\frac{94}{217}$, $\frac{12}{139}$; (*c*) $\frac{514}{8372}$, $\frac{1542}{981}$, $\frac{3}{44}$, $\frac{14}{23}$, $\frac{19}{41}$.

☆ ☆

The third type: calculation of the original value

§ 98 THE CAPITAL VALUE

Chiefly in work with money we meet examples where the original value is required, as the so-called capital value. A house brings in \$600 rent, what is its capital value? An apple tree produces on average \$15 worth of apples a year, what is the tree worth? The house and the tree bring in money, which is like interest, so they themselves can be regarded as capital: for example the \$600 could be regarded as 6% interest on the capital value of the house, or the \$15 as 5% interest on the capital value of the tree. The solution comes from the formula for percentages (§ 95):

$$g = \frac{100v}{p} = v \times \frac{100}{p}, \text{ that is, the}$$

RULE **Divide 100 times the p-value by the p-ratio.**

So the capital value of the house is $100 \times 600/6 =$ \$10000, that of the apple tree is $15 \times 100/5 = 15 \times 20 =$ \$300.

Note If the p-ratio is a stem-fraction, as $5\% = \frac{1}{20}$, *just multiply the p-value by the denominator of the stem-fraction.*

This works very well for the approximation of these

Exercises 25% is \$84; 14% is \$630; 16% is \$3·40.

☆ ☆

§ 99 SUMMARY

If we look back on the three fundamental types of sum, particularly the first two, which occur most frequently, we see that for a quick and accurate solution we chiefly need to know **two things**:

(1) **the key equation: $p\%$ is $\frac{p}{100}$ of the original value,**

(2) **the table of stem-fractions ⇄ p-ratios** (§ 97(*b*)),

which provide invaluable help in exact and approximate work and can also serve as tests in percentage work.

Everybody should master percentages, for they form one of the most important branches of everyday arithmetic. If you choose the methods given here, you will have an advantage over anyone working by the old rule of three: you will work more easily and accurately, and you will be much faster.

☆ ☆

CONCLUSION

§100

We have come to the end. If you have wandered through some of the land of arithmetic with me, you will have learnt, besides something useful, something important as well: the world of numbers is not dreary, certainly not as dreary as people often see it. Admittedly anybody who always keeps to the main road, who is never shown anything of the character of numbers and their wonderful interlacing, will always find arithmetic a drudgery.

For anyone who needs arithmetic for his business or her household accounts, the most important thing will be the convenience of the 'arithmetical tricks'; but for the youngster it should be the mental gain: getting rid of his anxiety about arithmetic, and awakening his enthusiasm. Believe me: boredom, which kills all enjoyment, is the cause of the trouble in arithmetic in nine cases out of ten. And if these 'arithmetical tricks' have contributed to turning the tables on boredom and killing it, they will have fulfilled their main task.

I would rather not take my leave of you with forefinger upraised, like a teacher, but with an amusing arithmetical game. To finish with, we will

catch the wren!

How do we do that? By ourselves, or preferably with others in competition, we will go hunting. A series of numbers is announced, and written down:

17 38 4 3 25 9.

They are to be *combined*, using any or all of the four operations of arithmetic, and whoever gets the *smallest whole number* has caught the wren.

The numbers can be *arranged in a different order*, but *fractions cannot* be used, nor can 0, except at the end, nor can any number *less* than 0.

We will try our luck once:

$$38+17=55,\ -25=30,\ +9=39,\ \div 3=13,\ -4=9.$$

Once more:

$$38-25=13,\ +17=30,\ -9=21,\ \div 3=7,\ -4=3.$$

Is 3 the wren?

ANSWERS

(t = true, f = false, r = remainder)

2 (*a*) 5, 1, 2 (*b*) 2, 2, 2, 2 (*c*) 5, 4, 9

4 14059716, 97406784, 28580485, 11712129

5 1740069, 378187, 0·03502, 82079

9 (*a*) 157 (*b*) t (*c*) 308 r 4 (*d*) t (*e*) 0·8

10 (*a*) 692 (*b*) $13415·65

13 (*a*) 1, 10, 8 (*b*) 8, 7, 4

15

	NT	*ET*	*Answer*	*Error*	*Right answer*
(*a*)	t	f	f	Position	1754505
(*b*)	f	t	f	1100 too many	3183545
(*c*)	t	t	t		564184
(*d*)	t	f	f	Position	0·08
(*e*)	t	f	f	Position	364·725

23 (*a*) 210 (*b*) 110 (*c*) 483 (*d*) 915

24 403792, 1·60658, 56·0606

25 (*a*) 2377·65 (*b*) $1797·41

26 55, 258, 356, 275, 7686

27 (*a*) 3211 (*b*) 43163 (*c*) $2·64

28 (*a*) 360, 630, 2240, 4180, 27340 (*b*) 435, 715, 1375, 24165, 3202855

30 (*a*) 475, 575, 850, 1400, 1625, 2100 (*b*) 2325, 3075, 3650, 6700
(*c*) 8175, 114600, 2432075 (*d*) 20000, 2291100, 12048, 59392

31 (*a*) 2100, 3300, 5100, 6900, 9300 (*b*) 18300, 1275, 3150, 4725

32 (*a*) 3125, 4125, 5500, 8375 (*b*) 10500, 30625, 42125, 7232750

33 (1) 72, 64·8, 28·8, 5·4 km/h (2) 1·8, 360, 828 km/h

34 (1) 264, 396, 792, 726, 1067, 638, 902
(2) 46277, 80135, 43208, 746845
(3) 825, 748, 968, 1001, 561, 1518
(4) 1571625, 93929, 5789597

35 (*a*) 330, 690, 960 (*b*) 7230, 8820, 26310
(*c*) 405, 795, 1305 (*d*) 4755, 1012935

37 (1) 2664, 4773, 3219, 8658
(2) 25641, 38517, 577866, 536685, $26 \times 111 = 2886$, $42 \times 111 = 4662$
(3) $72 \times 111 = 7992$, 12099, 56832, 15836

39 192, 306, 256, 266, 208

40 1980, 3306, 5329, 7308

41 184, 245, 584, 423, 496, 336

43 (*a*) 1271, 2788, 432, 1665 (*b*) 1881, 1836, 7138, 2028
(*c*) 8544, 4935, 3648 (*d*) 5904, 6864, 20746, 504

44 (*a*) 8370, 30895, 65255 (*b*) 61468, 52635

45 (*a*) 181390, 22572. (*b*) 375808, 209988 (*c*) 19205784
(*d*) 27·44477230

46 (*a*) 1344, 1935, 858, 3465, 7395 (*b*) 2046, 7719, 6336, 9025, 1944, 11124, 15494 (*c*) 861, 3111, 2925, 8775

47 (*a*) 616, 2112, 2025, 2349, 2009, 4136, 2035
(*b*) 1254, 3003, 11024, 21021 (*c*) 60024, 13216, 27221, 93016

49 (1) 224, 2025, 3009, 9016, 27221, 60021, 20244, 21624
(2*b*) 2256, 667, 2968, 7308, 15867, 66048

51 (*a*) 8924, 8184, 9025, 8036, 7663, 7176
(*b*) 990024, 976128, 958440, 99750100, 99570420, 99171450, 92308

52 11663, 12312, 15410, 18500, 1047370, 100740693

53 9964, 9592, 8346, 10580, 12240, 1007952

54 4899, 2484, 875, 9964, 62496, 756884

55 (*a*) 1806, 6162, 7921, 3844 (*b*) 154445, 622472, 24920064
(*c*) 42021, 370877, 847300 (*d*) 643986

56 (1) 137404305, 37695573, 2871068 (2) 2447304, 39627, 945376, 20658780 (3) 256473, 430760, 40548

59 (*a*) 729, 1089, 4096, 5776, 8464 (*b*) 13225, 13689, 64009, 161604

60 103684, 23658496, 7284281104

61 (1) 3136, 3481, 3721, 4096, 4761, 5184 (2) 2209, 1764, 1521, 1089, 784, 676 (3) 8649, 9604, 7569, 6724, 5929 (4) 10609, 11449, 13225

62 (1) 2025, 9025, 15625, 60025 (2) 65025, 245025, 105625, 342225
(3) 403225, 18225, 34225, 21170
(4) 3721, 8281, 121, 14641, 19881, 63001, 549081
(5) 992016, 99820081, 1022121, 1046529, 169744, 616225
(6) 180625, 680625, 2030625, 5880625, 525625, 1755625

63 (*a*) 266256, 304704, 239121, 180625, 25170289
(*b*) 255025, 40804, 826281, 169338169, 324648324, 3225501249

64 3136, 6241, 13456, 30276

69 (1) 16 r 2 or 16·4, 34 r 4 (·8), 65 r 2 (·4), 127405
(2) 12 r 17 (·68), 13, 14 r 8 (·32), 16 r 16 (·64), 51 r 3 (·12), 2960 r 21 (·84)
(3) 34, 257 r 2 (·016), 3666 r 28 (·224), 77918 r 124 (·992)
(4) 196480750 r 4, 39296150 r 4, 7859230 r 4

73 (*a*) 6294 r 13, 2447 r 61, 3916 r 25, 5340 r 25
(*b*) 31886338 r 3, 9964480 r 33, 8857316 r 9
(*c*) 0·46.., 0·452380.., 1·60606...

74 (*a*) 537 r 5 (*b*) 117 r 57, 1600 r 28, 489 r 12

75 (*a*) 10193 r 49, 4368 r 124 (*b*) 7795 r 159, 5568 r 34

77 (*a*) 735 = 3 × 5 × 7 × 7, 544 r 267 (*b*) 462 r 282
(*c*) 891 = 9 × 9 × 11, 721 r 405 (*d*) 421 r 180

78 (*a*) Divisible by 7 (*b*) Divisible by 13

79 (*A1*) Divisible by 17 (*A2*) Divisible by 19 (*B*) All divisible

82 (*a*) 111 r 26, 147 r 26, 140 r 5, 56 r 33 (*b*) 9944 r 20, 4218 r 64
(*c*) 0·063291.., 1·6756 (*d*) 426 r 755, 84 r 3329 = 84·83..

84 3260 r 7, 1157, 834 r 5

85 48 r 17, 672 r 377, 1·701

87 (*a*) $355 = 5 \times 71$, 1362 r 254 (*b*) $1368 = 8 \times 9 \times 19$, 256 r 973
(*c*) $7864 = 8 \times 983$, 3854 r 5377

92 (1) 41995, 126495, 40259, 7565, 22140975
(2)(*a*) 612, 1056, 672, 1406, 5642
(*b*) 10080, 344925, 5856, 225120, 2274615

95 (*a*) 96·69, \$586·38, 3885·84 (*b*) 2025, \$128·78, \$5510·40

97 (*a*) 50%, $33\frac{1}{3}\%$, $58\frac{1}{3}\%$, $42\frac{6}{7}\%$ (*b*) $16\frac{2}{3}\%$, $66\frac{2}{3}\%$, 43·3%,
8·6% (*c*) 6·1%, 157%, 6·8%, 60·8%, 46·3%

98 \$336; $\simeq 7 \times 630 =$ \$4410, exactly, \$4500; $\simeq 6 \times 3{\cdot}40 =$ \$20·40,
exactly, \$21·25

100 No, the wren is 0. One solution: $9 \times 3 = 27$, $-25 = 2$, $\times 17 = 34$, $+4 = 38$, $-38 = 0$. There may very well be other solutions giving 0 among the more than 730000 possibilities of combining the 6 numbers by means of the 4 signs.

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INDEX

The numbers refer to the sections. See also the table of contents

Abbreviated multiplication of decimals, 45(*b*)
Addition, 16–28
— mental, 21
— test for, 10
Approximation, 62(*a*), 90–92
Arithmetical progression, 23
Audit, 25
Auxiliary number (divisibility), 79 (*A*)

Beginning-squares, 61(*a*)
Boundary-division, 80–87
Boundary estimate, 90

Calculator, for the experienced, 17, 24, 32, 34(*d*), 39, 44, 45, 47, 49 (3), 51, 61(*C*), 62(*C*), 79, 83, 85, 86, 87, 92, 95
Capital value, 98
Carry-over, 22(*b*), 24, 70
Casting out 9-packets, 6(*b*)(1)
Catching the wren, 100
Chain of fractions, 90
Column multiplication, 49
Cross-multiplication, 42–48
— double-digit, 46
— of two-digit numbers, 42
— of numbers with more digits, 45
— with 10-complement, 47
— with *n*-fold 10-complement, 47
Cross-sum, 2

Decimals, 14, 45(*b*), 67, 68, 69, 73, 85, 90, 97(*c*)
— point rules, 90
Discarding multiples, 79 (*B II*)
Distance in km, 37
Divisibility by 7, 77, 79 (*A I*)
— by 11, 77, 78, 79 (*A II*, 1)
— by 13, 78, 79 (*A II*, 1)
— by any divisor, 79
Divisibility rules, 76–79
—'fabulous' number 1001, 78
— for any divisor, 79
— simple, 77
— with auxiliary number, 79 (*A*)
— with complement-remainder, 79 (*B*)
Division, 76–88
— by 5, 67
— by 25, 68, 75
— by 125, 69, 75
— by a boundary, 73
— by a prime number, 87
— by a unit-boundary, 66
— by a number near a unit-boundary, 85, 86
— of decimals, 90
— short, 70
— and subtraction at the same time, 24
—, test for, 9
Division cross (boundary-division), 82
Division methods, unusual
— factor-division, 71–79
— boundary division, 71, 80–87
Double of a two-digit number, 26 (2), 29 (*B*)
Double-digit, trick of the, 46
Double test, 15

ER = eleven-remainder, 13
Eleven-test, 13, 14

Error, decimal point, 5(*a*)
— of position, 5(*a*), 15
—, typing, 5(*b*)
Estimate, 1, 5(*a*), 38, 62(*a*), 89–92
— boundary-, 90
— with stem-fractions, 92, 95, 97
— with tricks, 91

Fabulous number 1001, 78
Factor-division, 71–75, 97(*c*)
— by a prime number, 87
Factorisation, 57, 76–79
Factorising and recombining, 57
Factor-multiplication, 57
Finger-multiplication, 100
Formula for difference of two squares, 54, 59, 60, 61(*D*), 62(*B*), 64
Fractions, comparison of, 96
—, enlarging and contracting, 97(*a*)
—, transformation to decimals, 73
'Funnel'-multiplication, 45(*b*)
— -squaring, 60

Group-vision, 29 (*A*)

Half of a number, 26 (2), 29 (*B*)
'Hooking up' (11-trick), 34, 37
100-complement, 26 (2)
100-stem-fractions, 92

Indices, 90

Key equation (percentages), 93

Large multiplication table, 39

Magic 11, 37, 78
Mathematician, for the, 8, 15, 55, 61(*D*)
Mental arithmetic, 21, 26, 61, 62, 90
— semi-, 22(*a*)
Mirror-numbers, 54
Miser, amusing story, 78
Money sums, 25, 27
Multiplication, 28–57
— by particular numbers, 28–38
— by 5, 28, 29 (*Ac*)
— by 9 and multiples of 9, 33
— by 11 and multiples of 11, 34
— by 15, 35
— by 25, 30
— by 37, 36
— by 75, 31
— by 111, 37
— by 125, 32
— by 375, 32
— of decimals, 90
— of mirror-numbers, 54
— of numbers ending in 1, 46 (1), 56 (3)
— of numbers ending in 5, 46 (2)
—, test for, 3–7
Multiplication, special methods
— by sliding, 45(*a*)
— column-, 49
— cross-, 42–48
— devices for written, 56
— factor-, 57
— of numbers near a boundary, 50–56
Multiplication table, 39–41
— large, 39
— medium, 41(*a*)
— small, 1, 39
— on the fingers, 41(*b*)

Negative numbers, work with, 45, 90
Neighbouring squares, 64
9-errors, 5*b*, 15
9-multiples, 5*b*, 33
9-packets, 6*b*
NR = nine-remainder, 8
Nine-test, 2–12
— certainty, 6(*a*), 14, 15
— name, 8(*a*)
— proof, 8(*b*)
— eleven-test (double-test), 15
Numbers ending in 1, 46 (1), 56 (3), 62(*B*)
— ending in 5, 46 (2), 62(*A*)
— near a boundary, 50–55, 57

Order of magnitude, 1, 5(*a*), 89
Original value, 94

Percentages, 93–99
— formulae, 95, 98
— fundamental types of sum, 94
— ground-value, 94, 98

Percentages, idea, 94
— key equation, 93, 99
— original value, 94, 98
— percentage-ratio, 93, 96, 97
— percentage-value, 95
— stem-fractions, p-ratios, 92, 95, 97, 99
Point rules for decimals, 90
Powers of 10, 61(*A*), 90
— estimate with, 90
— working with, 90
Prime numbers, 76, 87

Reading out large numbers, 29 (*Ad*)
Remainder calculation, 8(*b*), 13, 15
— -rule (factor-division), 74
Retaining numbers, 22(*a*), 70
Round numbers, 1, 90

Slide rule, 90, 97
Sliding method, 45(*a*)
Slope as a percentage, 96
Speeds, conversion, 33(*a*)
Splitting up a number, 21, 26 (1), 29 (*A*)
Squares, 58–65
— 1-squares, 62(*B*)
— 5-squares, 62(*A*)
— 25-squares, 62(*D*)
— of numbers in the fifties, 61(*B*)
— of numbers from 1 to 100 mentally, 61
— of numbers from 400 to 600, 63(*A*)
— of numbers consisting of threes or nines, 63(*B*)
— of numbers near a boundary, 61(*C*), 62(*C*)
— beginning-, 61(*A*)
— 'funnel', 60
— general methods, 57, 58
— 0-mirror, 63(*C*)
— neighbouring, 64
— particular, 62, 63
Stem-fractions, 92, 95, 97
Steps, working in, 19
Subtraction, 24–27
— 'Austrian', 24
— by adding up, 24
— mental, 26
— from a unit boundary, 27
— of several amounts from one, so-called audit, 25
—, test for, 11

Teachers of arithmetic, for, 6, 17, 26, 29 (*B*), 41(*a*), 56, 57, 59, 61(*A*), 70, 90
10-packets, 17
10-powers, 90
Test 1–12, 79 (*BIII*)
— addition, 10
— division, 9
— 11-, 13, 14
— for decimals, 14
— multiplication, 3
— 9-, 1–12
— 9-11- (double test), 15
— powers, 12
— roots, 12
— subtraction, 11, 25
— 37-, 79 (*BIII*)
1000-complement, 26 (2)
1000-stem-fractions, 92
1001-rule (fabulous number), 78
Two-digit numbers, working with, 21, 26, 29(*Ad*), 34, 39, 41, 43

Unit-boundary, 27, 55, 85, 90
— division by, 66, 79 (*B*)

Zero-factor (test), 7

CALCULATOR'S CUNNING

THE ART OF QUICK RECKONING

by

KARL MENNINGER

This book has become very well known in Germany, where it has gone through many editions.

Arithmetical tricks are devices for solving problems more easily, quickly—and surprisingly—than by ordinary methods. Numbers have a certain magic. Their wonderful properties have grown into thetheory of numbers, 'the queen of mathematics'. Those relations generally known as mathematical tricks belong in fact to elementary number theory. While one person will see behind them the wide and wonderfully intricate world of numbers, another will be satisfied by his delight in the tricks themselves and the sense of power over numbers whichthey give.

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